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On the instability of Taylor vortices

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It is known experimentally that laminar circular Couette flow between two concentric circular cylinders, the outer of which is fixed, becomes unstable when the speed of the inner cylinder is high enough. The flow is then replaced by a new circumferential flow with superimposed toroidal (or Taylor) vortices spaced periodically along the axis. At a higher speed still the new flow develops another instability, and is replaced by a flow in which the axially periodic vortices are simultaneously periodic travelling waves in the azimuth.

In the present paper an attack is made on the problem of instability of the Taylor-vortex flow against perturbations which are periodic both in the axial and azimuthal co-ordinates and, moreover, travel with some phase velocity in the latter. Subject to a number of assumptions and approximations, which are detailed in the paper, it is found that the Taylor-vortex flow is stable against perturbations with the same axial wavelength and phase, but unstable against perturbations differing in phase by $\frac{1}{2}\pi$. After instability the new flow no longer has planes separating neighbouring vortices, but has wavy surfaces travelling in the azimuth. This feature is in accord with much (though not all) of the experimental evidence.

The critical Taylor number (proportional to the square of the speed) at which the Taylor vortices become unstable is found theoretically to be about 8 % above the value for which Taylor vortices first appear. This must be compared with a value in the range 5–20 % for the experiments which our work models most closely. The azimuthal wave-number given a slight preference by theory is 1, in agreement with those experiments.

1. Introduction

Taylor (1923) in his classic paper showed both theoretically and experimentally that the laminar circumferential flow (Couette flow) between concentric rotating cylinders becomes unstable if the speed of the inner cylinder is increased beyond a certain critical value. He observed that the instability yielded a steady secondary motion in the form of toroidal vortices (Taylor vortices) spaced regularly along the axis of the cylinders. Theoretically, the critical speed can be predicted by considering the linearized problem for the stability of Couette flow with respect to axisymmetric disturbances. This leads to an eigenvalue problem for the Taylor number T (based on the speed of the inner cylinder) which is a function of the parameters $\mu = \Omega_2/\Omega_1$ and $\eta = R_1/R_2$ describing the basic velocity and geometry up to scale factors, and the dimensionless axial wave-number λ of the disturbance. Here Ω_1, Ω_2 and R_1, R_2 are the angular velocities and radii of the inner and outer cylinders respectively.

Since Taylor's original work, there has developed a considerable body of literature dealing with the mathematical eigenvalue problem and with experimental measurements of the transition boundary. Much of this work, including generalizations of the Taylor stability problem, is discussed by Chandrasekhar (1961); also see a brief survey paper by Di Prima (1963). It is probably safe to say that both experimentally and theoretically the transition boundary from Couette flow to Taylor-vortex flow and the dependence of the critical Taylor number on the parameters μ and η is well understood. A note of caution is necessary, however, as recently Krueger, Gross & Di Prima (1966) have shown that for $\mu < -1$ approximately, and, for a wide range of values of η, non-axisymmetric disturbances become unstable at Taylor numbers slightly lower than the critical Taylor number for axisymmetric disturbances. Apparently the instability leads to a weak spiral vortex motion as observed by Coles (1965, p. 399) and Snyder & Karlsson (1965).

In the present paper we will be concerned with the development of finite amplitude motions for $T > T_c$, where T_c is the critical value of T at which Couette flow becomes unstable. Further, we shall restrict our attention to the case in which the cylinders (supposed infinitely long) rotate in the same direction or in which the outer cylinder is at rest ($\mu \geqslant 0$), and the gap between the cylinders is small compared to a typical radius ($\eta \to 1$). According to linear theory, when the basic flow is unstable the disturbance grows exponentially with time for $T > T_c$. However, as Taylor observed, it is known that a definite equilibrium vortex motion is attained. Moreover, the circulation in the vortices is a function of $T - T_c$. According to Taylor, 'A moderate increase in the speed of the apparatus merely increased the vigour of the circulation in the vortices without altering appreciably their spacing or position, but a large increase caused the symmetrical motion to break down into some kind of turbulent motion...'.

In addition to Taylor's work, Coles (1960, 1965), Schwarz, Springett & Donnelly (1964), Nissan, Nardacci & Ho (1963), and Schultz-Grunow & Hein (1956) have made experimental observations of Taylor vortices for T increasing beyond T_c for the case $\mu \geqslant 0$. Their observations appear to confirm Taylor's observations for moderate values of $T > T_c$. On the other hand, for T sufficiently large, the vortices assume a wavy form in the circumferential direction and have a certain wave velocity in that direction. With increasing speed different wavy motions develop, until at considerably higher speeds small irregularities begin to appear and the flow becomes turbulent.

A particularly comprehensive and brilliant account of the flow development in one apparatus (with $\eta = 0{\cdot}88$, $\mu = 0$) has been given by Coles (1965). While his apparatus had a small length-to-gap ratio, being able to accommodate only 30 Taylor vortices compared with G. I. Taylor's 400, it is still pertinent to summarize briefly his observations. Let a given flow be denoted by m/n where m is the number of Taylor vortices and n is the number of azimuthal waves. Then Coles found at a rising sequence of quite definite (and repeatable) speeds of the inner cylinders the sequence of states 28/0 (Taylor vortices), 28/4 (wavy vortices at about $1{\cdot}5T_c$), 24/5, 22/5, 22/6,... In all cases for which $n \neq 0$, the boundaries between neighbouring cells were wavy. The angular wave speed was about equal to the average angular velocity between the cylinders at the first appearance of the wavy vortices, but decreased with increasing speed to about 0·34 of the inner cylinder's angular velocity. In addition, he observed that in the range of speeds for which doubly-periodic flows were possible, different states of motion could be attained at the same final speed—the state depending upon the manner in which the final speed was reached. In Coles' words, 'the experimental fact is that the steady Couette flow of a given fluid in a given apparatus is not uniquely determined by the speed of rotation...'.

While Coles observed the transition from Taylor-vortex flow to wavy-vortex flow to be a 28/0 to 28/4 transition, Schwarz *et al.* (1964) using an apparatus which could accommodate approximately 260 Taylor cells ($\eta = 0{\cdot}95$, $\mu = 0$) have apparently observed a transition to a non-axisymmetric mode with azimuthal wave-number 1 at a Taylor number 3–8 % above critical. The mode appeared to be a subtle modification of the Taylor-vortex mode and moved with an angular velocity nearly equal to the average angular velocity of the basic flow. In addition, the mode appeared to have a regular vortex spacing in the axial direction with planes, perpendicular to the axis and separating neighbouring vortices, on which the axial component of velocity vanished. No stable modes of this type, $\exp[i(m\theta - \omega t)]$, for $m > 1$ were observed. As T was increased the circulation in the $m = 1$ mode grew more vigorous and the axial form became more distorted with the vortex spacing sinusoidal in time. This evolution took place over a range of T but, as explained by Schwarz *et al.*, at a T of about 20 % above T_c appeared to be complete, in the sense that it had a form which could definitely be described as of wavy-vortex type.

In recent years attempts have been made to compute the Taylor-vortex motion. An energy balance integral method has been used (Stuart 1958) which takes account of the distortion of the mean motion by the disturbance, and gives a finite non-zero equilibrium amplitude for the secondary circulation in the vortices. Following this work an expansion procedure (Stuart 1960; Watson 1960) has been used (Davey 1962) to compute the amplitude and form of the Taylor-vortex motion for a range of speeds above critical. This analysis is valid to second-order terms in the amplitude (though it involves consideration of third-order terms), and takes account of the distortion of the mean motion, of the generation of the first harmonic of the fundamental, and of the spatial distortion of the fundamental. Good agreement with experimentally-measured torque data, particularly for T near T_c, is found in both analyses. It is indicated in the later paper (Davey 1962) why the energy-balance method is so effective at equilibrium.

In the above work it is assumed that the wavelength of the vortices in the axial direction is the same as that predicted by linear theory. Even though it is found experimentally that the variation of wavelength with increasing T is not large (Donnelly & Schwarz 1965), it has been suggested by Meyer (1966) that a suitably chosen variation of wavelength may lead to better agreement with experiment. He has carried out an extensive numerical calculation of the Taylor-vortex flow using a time-dependent finite difference procedure, and has found agreement with Davey's calculations for the fixed wave-number predicted by linear theory. But by suitably varying the wave-number as T increased, he could obtain agreement with the experimental torque data over a much wider range than obtained by Davey (1962) in the small-gap case. However, the required variation in wavelength was much larger than observed experimentally. Meyer also suggested that a possible mechanism for the transition from Taylor-vortex flow to wavy-vortex flow is a shear instability of the circumferential velocity profile which has a large variation between neighbouring cells. This suggestion will be discussed in § 6. These regions of high shear in the circumferential velocity at the boundaries of neighbouring cells have also been noted experimentally by Snyder & Lambert (1966). Finite difference procedures for computing the Taylor-vortex motion have also been considered by Capriz, Ghelardoni & Lombardi (1964, 1966).

In our analysis of the growth of Taylor vortices and their instability, we shall consider the interaction of two axisymmetric disturbances of differential axial phases with two non-axisymmetric disturbances of different axial phases. Follow-a method described elsewhere (Stuart 1961), we derive a system of four non-linear equations for the amplitudes of the fundamental disturbances as functions of time. The coefficients in the amplitude equations are functions of the parameters of the problem, namely (i) μ and η, which describe the laminar velocity and geometry; (ii) the axial and circumferential wave-numbers of the disturbance; and (iii) the Taylor number. These coefficients can be determined in a systematic manner by solving a set of linear ordinary differential equations. Possible equilibrium states of our mathematical model (the non-linear amplitude equations), the stability of the equilibrium states, and the transition from one equilibrium state to another as the Taylor number increases, will be discussed.

The present model is sufficiently general to admit equilibrium solutions corresponding to the mode observed by Schwarz *et al.* (1964), and also the more complex wavy-vortex modes, and to give results concerning their stability. Finally, we mention that this work illustrates the concept of successive instabilities suggested by Landau (1944) in the sense that the model includes (i) a stability analysis of the Taylor-vortex flow with respect to non-axisymmetric disturbances and (ii) the possible states of motion that might result from an instability of the Taylor-vortex flow.

In the next section the small-gap disturbance equations are derived. In § 3 a Fourier analysis of the disturbance and suitable expansions of the Fourier components in powers of the amplitudes of the four fundamental disturbances are discussed. Sections 4 and 5 deal with possible solutions of the non-linear amplitude equations and their stability. Finally, in § 6 we discuss the relevance of the theoretical results to experimental observations; our conclusions are summarized in § 7.

2. The disturbance equations

Let r, θ, z denote cylindrical-polar co-ordinates, and let u_r, u_θ, u_z denote the components of velocity in the r, θ and z directions respectively. Consider two infinitely-long concentric circular cylinders with the z axis as their common axis, with radii R_1 and R_2 ($>R_1$), and rotating with angular velocities Ω_1 and Ω_2 respectively. The equations of motion for a viscous, incompressible fluid admit the exact steady solution

$$u_r = u_z = 0, \quad u_\theta = V(r) = Ar + (B/r), \tag{2.1}$$

where A and B are constants chosen so that (2.1) satisfies the boundary conditions at $r = R_1$ and $r = R_2$. In order to study the stability of this flow we superimpose a general disturbance on this basic solution and write, for example

$$u_\theta = V(r) + v'(r, \theta, z, t). \tag{2.2}$$

Substituting in the Navier–Stokes equations of motion and in the equation of continuity (e.g. Whitham 1963), we obtain a system of four non-linear partial differential equations for v' and for the perturbations u', w' and p' in u_r, u_z, and the pressure respectively.

In the present analysis we shall restrict our attention to the 'small-gap' case, in which the gap $d = R_2 - R_1$ is so small compared to the mean radius $R_0 = \frac{1}{2}(R_1 + R_2)$ that terms $O(d/R_0)$ can be neglected. The derivation of the small-gap equations is essentially the same as for the classical Taylor problem except that now we must also consider terms involving differentiation with respect to the circumferential co-ordinate θ. The proper scaling for θ has been discussed by Krueger (1962), by Bisshopp (1963) and by Krueger *et al.* (1966) in their analyses of the linear stability problem for non-axisymmetric disturbances. Briefly, the reasoning is as follows. Consider the second momentum equation

$$\frac{\partial u_\theta}{\partial t} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \ldots = \nu\left(\frac{\partial^2 u_\theta}{\partial r^2} + \ldots\right)\ldots, \tag{2.3}$$

where ν is the kinematic viscosity. Letting Ω_0 denote a reference angular velocity, and scaling t in units d^2/ν and u_θ in units of $R_0\Omega_0$, we see that the second term in (2.3) is of apparent scale $(\Omega_0 d^2/\nu)$ as compared with the other terms in the equation. Recalling that, for the classical Taylor problem in the limit $d/R_0 \to 0$, the dimensionless combination $(\Omega_0 R_0 d/\nu)(d/R_0)^{\frac{1}{2}}$ is kept fixed,† we see that, if the second term in equation (2.3) is to be retained, then $\partial/\partial\theta$ must introduce a factor $(R_0/d)^{\frac{1}{2}}$. In that case $(\Omega_0 d^2/\nu)\partial/\partial\theta$ has a scale $(\Omega_0 R_0 d/\nu)(d/R_0)^{\frac{1}{2}}$, and the term must be retained.

For the linear stability problem it is often convenient to obtain a single sixth-order equation for the circumferential perturbation velocity v', by using the con-

† The appearance of $(d/R_0)^{\frac{1}{2}}$ represents the curvature effect.

tinuity equation, and the θ and z momentum equations to eliminate w', p' and u' respectively. For the non-linear equations this is no longer possible. It is still convenient, however, to follow a similar procedure to obtain an equation whose *linear* part has the above form. At the same time we obtain subsidiary equations which relate u' and w' to v' (the pressure need not concern us here). Letting $d/R_0 \to 0$ with $\Omega_0^2 R_0 d^3/\nu^2$ and $\nu\theta/\Omega_0 d^2$ fixed, we have

$$\left.\begin{aligned} LMLv - T\Omega_l(x)\frac{\partial^2 v}{\partial \zeta^2} &= -\frac{1}{\alpha}\frac{\partial^2 P_1}{\partial \zeta^2} - \frac{1}{\alpha}\left\{LM + \alpha\frac{\partial^2}{\partial x\,\partial\phi}\right\}P_2 + \frac{1}{\alpha}\frac{\partial^2 P_3}{\partial x\,\partial\zeta}, \\ Lv - u &= \frac{1}{\alpha}P_2, \\ \frac{\partial u}{\partial x} - \alpha\frac{\partial v}{\partial \phi} + \frac{\partial w}{\partial \zeta} &= 0. \end{aligned}\right\} \tag{2.4}$$

Here we have chosen $\Omega_0 = \frac{1}{2}(\Omega_1 + \Omega_2) = \Omega_1\frac{1}{2}(1+\mu)$ where $\mu = \Omega_2/\Omega_1$, and have introduced the dimensionless variables and operators

$$\left.\begin{aligned} &r = R_0 + dx, \quad z = \zeta d, \quad \theta = \frac{\Omega_0 d^2}{\nu}\phi, \quad t = \frac{d^2}{\nu}\tau, \quad T = \frac{-4A\Omega_0 d^4}{\nu^2}, \\ &V(r) = R_0\Omega_0\Omega_l(x), \quad \Omega_l(x) = (1-\alpha x), \quad \alpha = 2\frac{1-\mu}{1+\mu}, \\ &v' = R_0\Omega_0 v, \quad u' = -\frac{\nu}{\alpha d}u, \quad w' = -\frac{\nu}{\alpha d}w, \\ &M = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \zeta^2}, \quad L = M - \frac{\partial}{\partial \tau} - \Omega_l(x)\frac{\partial}{\partial \phi}. \end{aligned}\right\} \tag{2.5}$$

The parameter T is often called the Taylor number. Moreover

$$\left.\begin{aligned} P_1 &= u\frac{\partial u}{\partial x} - \alpha v\frac{\partial u}{\partial \phi} + w\frac{\partial u}{\partial \zeta} - \tfrac{1}{2}\alpha T v^2, \\ P_2 &= u\frac{\partial v}{\partial x} - \alpha v\frac{\partial v}{\partial \phi} + w\frac{\partial v}{\partial \zeta}, \\ P_3 &= u\frac{\partial w}{\partial x} - \alpha v\frac{\partial w}{\partial \phi} + w\frac{\partial w}{\partial \zeta}. \end{aligned}\right\} \tag{2.6}$$

If the non-linear terms P_1, P_2 and P_3 are neglected, then equations (2.4) reduce, with slightly different notation, to the linearized equations for stability with respect to non-axisymmetric disturbances considered by Krueger *et al.* (1966). The choice of $\Omega_0 = \Omega_1\frac{1}{2}(1+\mu)$ is a convenient scale if the cylinders rotate in the same direction; however, if the cylinders rotate in the opposite direction, a more appropriate scale for $\Omega(r)$ is Ω_1. In this case the only changes that are necessary in equations (2.4), (2.5) and (2.6) are the replacement of T, $\Omega_l(x)$ and α by $-4A\Omega_1 d^4/\nu^2$, $\frac{1}{2}(1+\mu) - \alpha x$ and $1-\mu$ respectively. The choice of scale for u' and w' relative to v' is the natural choice from the equations of motion as indicated by Davey (1962). It should be emphasized that equations (2.5) are small-gap equations, obtained from the full equations by letting $d/R_0 \to 0$ with the independent variables x, ϕ, z, τ, the dependent variables u, v, w, and the parameters μ and T held fixed.

Finally, in deriving equations (2.4) it has been assumed that the components of the disturbance depend upon z. For harmonic components which are independent of z, those equations are inconvenient and do not determine w. However, the equation for w separates from those for u and v and the system of equations (2.4) is conveniently replaced by the equivalent system

$$\left.\begin{aligned}&\left(\frac{\partial}{\partial x}L\frac{\partial}{\partial x}-\alpha\frac{\partial^2}{\partial x\,\partial\phi}\right)u=-\frac{\partial^2 P_2}{\partial x\,\partial\phi},\\&Lw=-(1/\alpha)P_3,\\&\frac{\partial u}{\partial x}-\alpha\frac{\partial v}{\partial\phi}=0,\end{aligned}\right\}\tag{2.7}$$

with the understanding that $\partial/\partial\zeta$ is to be set equal to zero wherever it appears in the operator L and the expressions for P_2 and P_3.

The requirement of no slip at the boundaries leads to the conditions

$$u=v=w=0\tag{2.8}$$

at $x = \pm\frac{1}{2}$. Furthermore, equations (2.8) imply for the system (2.4) that

$$\frac{\partial u}{\partial x}=\frac{\partial^2 v}{\partial x^2}=\frac{\partial(Lv)}{\partial x}=0\tag{2.9}$$

at $x = \pm\frac{1}{2}$; while for the system (2.7) we obtain only the first of equations (2.9).

3. An expansion procedure

For a given value of $\mu < 1$ the velocity distribution (2.1) is unstable† according to linear theory for Taylor numbers greater than a critical value T_c which depends upon μ. As mentioned earlier, for $\mu \geqslant 0$ and T slightly greater than T_c, the instability leads to a new motion composed of toroidal vortices spaced regularly in the axial directions and superimposed on a circumferential motion. With increasing T this laminar motion becomes unstable, the second instability apparently leading to a 'wavy' vortex motion.

Consider, first, the linearized problem for the stability of Couette flow. The critical Taylor number occurs for an axisymmetric disturbance, so we look for a solution of the linearized equations corresponding to equations (2.4) of the form $v(x,\tau,\zeta) = f(x)\cos\lambda\zeta.e^{a\tau}$. This leads to an eigenvalue problem for $a(T,\mu,\lambda)$. The critical value of T and the corresponding critical value of λ, λ_c, are determined by the requirement that T_c be the minimum value of T over all positive λ for which there exists solutions with $a = 0$ but not $a > 0$. For $\mu = 0$ it is known (Davey 1962, p. 363), that

$$T_c \cong 1694{\cdot}95,\quad \lambda_c \cong 3{\cdot}13.\tag{3.1}$$

For $T > T_c$ linear theory predicts that the Taylor-vortex disturbance will grow exponentially. In fact, however, as a disturbance of axial wave number λ

† For $\mu > 1$ it has been rigorously shown that Couette flow is stable to axisymmetric disturbances (see Chandrasekhar 1961, §70). A corresponding proof for non-axisymmetric disturbances has not yet been given.

grows, non-linear effects become important, altering the exponential growth so that an equilibrium state is attained. As described in the introduction, this non-linear problem has been studied by Stuart (1958) and Davey (1962) for T slightly greater than T_c.

Here, we shall extend this earlier work on the growth of the toroidal vortices by studying their instabilities and the form of motions consequent upon the instabilities. Thus we consider the interaction of a Taylor-vortex disturbance which is periodic in the axial direction only, with a non-axisymmetric disturbance, which is periodic in both the axial and the circumferential directions. Both disturbances are assumed to have the same axial wavelength $2\pi/\lambda$ (the variation of λ with T is slight in experiment for some range of $T > T_c$, as shown by Donnelly & Schwarz 1965); however, to allow for phase shifts in the axial direction as the motions develop, we consider the interaction of modes proportional to $\cos\lambda\zeta$ and to $\sin\lambda\zeta$.† Through the non-linear terms they interact and give rise to a mean velocity and to higher harmonics.

The general Fourier series representation for a function which is periodic of period $2\pi/\lambda$ in ζ and $2\pi/k$ in ϕ is

$$v(x,\zeta,\phi,\tau) = \sum_{q=-\infty}^{\infty}\left\{v_{0q}(x,\tau)+\sum_{n=1}^{\infty} v_{cnq}(x,\tau)\cos n\lambda\zeta+v_{snq}(x,\tau)\sin n\lambda\zeta\right\}e^{iqk\phi}, \tag{3.2}$$

where k is related to m of § 1 by $k = m\Omega_0 d^2/\nu$. Here the c or s subscript denotes whether the function is the coefficient of a cosine or sine; the first (n) and second (q) number suffixes refer to the harmonics of the axial and azimuthal wavelengths respectively; and the coefficients corresponding to q negative are the complex conjugates of the corresponding coefficients for q positive.‡ The terms

$$v_{c10}(x,\tau)\cos\lambda\zeta, \quad v_{s10}(x,\tau)\sin\lambda\zeta, \tag{3.3}$$

which represent Taylor vortex-modes, and

$$v_{c11}(x,\tau)\cos\lambda\zeta\,.\,e^{ik\phi}, \quad v_{s11}(x,\tau)\sin\lambda\zeta\,.\,e^{ik\phi}, \tag{3.4}$$

which represent non-axisymmetric modes, are the four fundamentals whose interaction we wish to consider. The additional terms in (3.2) represent harmonics and the mean-motion corrections, and arise from the interaction.

When we substitute from equation (3.2) for v, together with similar expressions for u and w, in the systems of equations (2.4) and (2.7) and separate out the various harmonics we obtain a system of infinitely-many partial differential equations (coupled non-linearly) for the functions $v_{c10}(x,\tau)$, $u_{c10}(x,\tau)$, $w_{c10}(x,\tau)\ldots$. By using an expansion procedure devised by Watson (1960) and Stuart (1961), this system of coupled partial differential equations can be reduced to a system of linear ordinary differential equations which can be solved in succession.

† For the Taylor-vortex motion u and v are symmetric functions of ζ while w, by the continuity equation, is anti-symmetric; consequently for that motion it was sufficient to use cosine and sine Fourier series respectively (see Davey 1962).

‡ Thus $v_{cn0}(x,\tau)$, $v_{sn0}(x,\tau)$ and $v_{00}(x,\tau)$ are necessarily real, while the other terms in the series may be expected to be complex-valued.

Briefly, the idea is as follows. Associated with the fundamental $v_{c10}(x,\tau)\cos\lambda\zeta$ is an amplitude $A_c(\tau)$. According to linear theory if we substitute

$$v_{c10}(x,\tau) = A_c(\tau) f_0(x),$$

we find $A_c(\tau) \sim \exp[a_{c0}\tau]$; hence, $dA_c/d\tau = a_{c0}A_c$ where a_{c0}, the amplification rate, is positive for $T > T_c$. Note that a_{c0} depends upon μ, λ and T. With increasing A_c this linear relation will cease to hold, the right-hand side of the equation for $dA_c/d\tau$ being replaced by $a_{c0}A_c$ plus higher order terms. Similarly with v_{s10}, v_{c11}, v_{s11} we associated the amplitudes A_s, B_c, B_s, respectively. While A_c and A_s are real-valued functions of τ, B_c and B_s will in general be complex-valued functions. When the fundamentals interact they give rise to the first harmonics of the fundamentals and to a mean-motion correction. These effects are represented (to first order) by quadratic terms in A_c, A_s, B_c, B_s. In turn, these terms react with the fundamentals and lead to distortions of the spatial form of the fundamentals and, moreover, force higher harmonics; such effects are represented by cubic terms in the A_c, A_s, B_c, B_s. The process cascades to higher amplitudes but, in a sense which will be discussed later, we may consider a termination of the series at this stage.

Thus we expand the velocities in suitable powers and products of the amplitudes $A_c(\tau)$, $A_s(\tau)$, $B_c(\tau)$, $B_s(\tau)$, the coefficients being functions of x. Correspondingly, the amplitudes satisfy a system of four non-linear first-order ordinary differential equations. It is not difficult, though it is rather lengthy, to show that the correct expansions are as follows.

The four fundamentals.

$$\left.\begin{aligned} v_{c10}(x,\tau) &= A_c f_0 + A_c^3 f_1 + A_c A_s^2 f_2 + A_c|B_c|^2 f_3 + A_c|B_s|^2 f_4 \\ &\qquad\qquad + A_s B_c \tilde{B}_s f_5 + A_s \tilde{B}_c B_s f_6 + \ldots, \\ u_{c10}(x,\tau) &= A_c f_{20} + \ldots, \\ w_{s10}(x,\tau) &= A_c f_{30} + \ldots. \end{aligned}\right\} \tag{3.5}$$

$$\left.\begin{aligned} v_{c11}(x,\tau) &= B_c h_0 + B_c|B_c|^2 h_1 + B_c|B_s|^2 h_2 + B_c A_c^2 h_3 \\ &\qquad\qquad + B_c A_s^2 h_4 + B_c A_c A_s h_5 + \tilde{B}_c B_s^2 h_6 + \ldots, \\ u_{c11}(x,\tau) &= B_c h_{20} + \ldots, \\ w_{s11}(x,\tau) &= B_c h_{30} + \ldots. \end{aligned}\right\} \tag{3.6}$$

Here the f's and h's are functions of x alone, and a tilde denotes a complex conjugate. The expansions for v_{s10}, u_{s10}, w_{c10} and v_{s11}, u_{s11}, w_{c11} respectively, are the same as those just given with A_c and A_s and B_c and B_s interchanged, and with the f's replaced by g's and the h's replaced by l's. Notice that the leading terms in the expansions are first order in the amplitudes, and that the corrections are third order.

The mean motion.

$$\left.\begin{aligned} v_{00}(x,\tau) &= A_c^2 F_1 + A_s^2 F_2 + |B_c|^2 F_3 + |B_s|^2 F_4 + \ldots, \\ w_{00}(x,\tau) &= A_c A_s G_1 + B_c \tilde{B}_s G_2 + \tilde{B}_c B_s G_3 + \ldots, \quad u_{00} \equiv 0. \end{aligned}\right\} \tag{3.7}$$

The F's and G's are functions of x alone. The fact that $u_{00}(x,\tau) \equiv 0$ follows from the continuity equation and the boundary conditions.

The first harmonics.

$$\left.\begin{aligned} v_{c20}(x,\tau) &= A_c^2 m_1 + A_s^2 m_2 + |B_c|^2 m_3 + |B_s|^2 m_4, \\ v_{s20}(x,\tau) &= A_c A_s n_1 + B_c \tilde{B}_s n_2 + \tilde{B}_c B_s n_3, \\ v_{c22}(x,\tau) &= B_c^2 p_1 + B_s^2 p_2, \quad v_{s22}(x,\tau) = B_c B_s q_1, \\ v_{c21}(x,\tau) &= A_c B_c r_1 + A_s B_s r_2, \quad v_{s21}(x,\tau) = A_c B_s s_1 + A_s B_c s_2. \end{aligned}\right\} \tag{3.8}$$

The series have been truncated at quadratic terms, and the functions m, n, p, q, r, s depend on x only. The expansions for u_{c20}, w_{s20} are the same as that for v_{c20} with $m_1, \ldots, m_4$ replaced by $m_{21}, \ldots, m_{24}$ and $m_{31}, \ldots, m_{34}$, respectively. The expansions for u_{s20}, $w_{c20}, \ldots, u_{s21}$, w_{c21} are determined similarly. In addition, for the harmonic components which are independent of ζ,

$$\left.\begin{aligned} v_{01}(x,\tau) &= A_c B_c t_1 + A_s B_s t_2, & v_{02}(x,\tau) &= B_c^2 y_1 + B_s^2 y_2, \\ u_{01}(x,\tau) &= A_c B_c t_{21} + A_s B_s t_{22}, & u_{02}(x,\tau) &= B_c^2 y_{21} + B_s^2 y_{22}, \\ w_{01}(x,\tau) &= A_c B_s z_1 + A_s B_c z_2, & w_{02}(x,\tau) &= B_c B_s z_3, \end{aligned}\right\} \tag{3.9}$$

to second order in amplitude. The functions t, y, z depend on x only.

The amplitude equations. For the above expansions to be consistent with equations (2.4) and (2.7), the amplitude functions $A_c(\tau)$, $A_s(\tau)$, $B_c(\tau)$ and $B_s(\tau)$ must satisfy a system of ordinary non-linear differential equations of the form

$$\left.\begin{aligned} dA_c/d\tau &= a_{c0} A_c + a_{c1} A_c^3 + a_{c2} A_c A_c^2 + a_{c3} A_c |B_c|^2 + a_{c4} A_c |B_s|^2 \\ &\qquad + a_{c5} A_s B_c \tilde{B}_s + a_{c6} A_s \tilde{B}_c B_s + \ldots, \\ dB_c/d\tau &= b_{c0} B_c + b_{c1} B_c |B_c|^2 + b_{c2} B_c |B_s|^2 + b_{c3} B_c A_c^2 + b_{c4} B_c A_s^2 \\ &\qquad + b_{c5} B_s A_c A_s + b_{c6} \tilde{B}_c B_s^2 + \ldots. \end{aligned}\right\} \tag{3.10}$$

The equation for A_s is similar to the equation for A_c with the a_c's replaced by a_s's and A_c and A_s and B_c and B_s interchanged; similarly the equation for B_s is the same as that for B_c with the b_c's replaced by b_s's and A_c and A_s and B_c and B_s interchanged. The parameters $a_{c0}, \ldots, a_{c6}$, $a_{s0}, \ldots, a_{s6}$, $b_{c0}, \ldots, b_{c6}$, $b_{s0}, \ldots, b_{s6}$ are functions of μ, λ, k, T. They, as well as the functions $f_0(x), \ldots, z_3(x)$, can be determined in a systematic manner, which we now discuss.

Substituting the series expansions (3.2) and (3.5)–(3.9) for v_{c10}, u_{c10}, $w_{s10}, \ldots$, $w_{02}(x,\tau)$ in equations (2.4) and (2.7), using equations (3.10), and equating coefficients of A_c, A_c^3, $A_c^2 A_s$, etc., we obtain equations for the functions $f_0(x)$, $f_{20}(x)$, $f_{30}(x)$, $f_1(x), \ldots, z_3(x)$. While it is not practicable to write out all of these equations, it is helpful to record a few of them in order to illustrate the method of solution. First, however, we define the following operators:

$$\begin{aligned} N(\lambda, a_{c0}, k) &= D^2 - \lambda^2 - a_{c0} - ik\Omega_l(k), \\ M(\lambda, a_{c0}, k, T) &= N(\lambda, a_{c0}, k)\,(D^2-\lambda^2) N(\lambda, a_{c0}, k) + \lambda^2 T \Omega_l(x). \end{aligned}$$

First-order terms. In the equations for the coefficients of $\cos\lambda\zeta$, $\sin\lambda\zeta$, $\exp[ik\phi]\cos\lambda\zeta$, and $\exp[ik\phi]\sin\lambda\zeta$, we have respectively

$$A_c\colon\ M(\lambda, a_{c0}, 0) f_0 = 0, \quad f_{20} = N(\lambda, a_{c0}, 0) f_0, \quad f_{30} = -\lambda^{-1} D f_{20}; \tag{3.11}$$

$$A_s\colon\ M(\lambda, a_{s0}, 0) g_0 = 0, \quad g_{20} = N(\lambda, a_{s0}, 0) g_0, \quad g_{30} = \lambda^{-1} D g_{20}; \tag{3.12}$$

$$B_c\colon\ M(\lambda, b_{c0}, k) h_0 = 0, \quad h_{20} = N(\lambda, b_{c0}, k) h_0, \quad h_{30} = \lambda^{-1}(-Dh_{20} + ik\alpha h_0); \tag{3.13}$$

$$B_s\colon\ M(\lambda, b_{s0}, k) l_0 = 0, \quad l_{20} = N(\lambda, b_{s0}, k) l_0, \quad l_{30} = \lambda^{-1}(Dl_{20} - ik\alpha l_0). \tag{3.14}$$

From equations (3.6) and (3.13), with some simplifications, the boundary conditions associated with the equation for h_0 are

$$h_0 = D^2h_0 = N(\lambda, b_{c0}, k)Dh_0 = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}. \tag{3.15}$$

The boundary conditions for l_0, g_0 and f_0 are the same as (3.15) except for l_0 replace b_{c0} by b_{s0}, for g_0 replace b_{c0} by a_{s0} and set $k = 0$, for f_0 replace b_{c0} by a_{c0} and set $k = 0$.

The homogeneous linear differential equation for f_0 with the associated homogeneous boundary conditions determines an eigenvalue problem for a_{c0} as a function of λ, μ and T, namely the linear stability problem for

$$v(x, \tau, \zeta) = f_0(x) \exp[a_{c0}\tau] \cos\lambda\zeta.$$

The parameter a_{c0} is clearly the amplification rate which is equal to or greater than zero for $\lambda = \lambda_c$ and $T = T_c$ or $T > T_c$ respectively. The eigenvalue problem for a_{c0} can be solved numerically for given values of μ, λ and T and the corresponding eigenfunction $f_0(x)$ tabulated, as has been done by Davey (1962) and Krueger *et al.* (1966). Similarly, equations (3.12), (3.13) and (3.14) with the appropriate boundary conditions determine eigenvalues problems for $a_{s0}(\lambda, \mu, T)$, $b_{c0}(\lambda, \mu, k, T)$ and $b_{s0}(\lambda, \mu, k, T)$. These eigenvalues and the corresponding eigenfunctions can be computed numerically. For a given disturbance, i.e. given values of λ and k, and for fixed μ and T, we are interested in the most highly amplified (or least damped) modes, that is, in the solutions of (3.11)–(3.14) corresponding to the largest eigenvalues a_{c0} and a_{s0}, and the eigenvalues b_{c0} and b_{s0} with the largest real part† respectively. It is clear from equations (3.11)–(3.14) that the s and c eigenvalues and eigenfunctions (appropriately normalized) are related by

$$\left.\begin{array}{lll} a_{s0} = a_{c0} = a_0, & b_{s0} = b_{c0} = b_0, & \\ g_0 = f_0, & g_{20} = f_{20}, & g_{30} = -f_{30}, \\ l_0 = h_0, & l_{20} = h_{20}, & l_{30} = -h_{30}. \end{array}\right\} \tag{3.16}$$

Second-order terms. In addition to corrections to the mean flow, second-order terms arise from the following:

$$\cos 2\lambda\zeta,\ \sin 2\lambda\zeta,\ \exp[i2k\phi]\cos 2\lambda\zeta,\ \exp[i2k\phi]\sin 2\lambda\zeta,$$
$$\exp[ik\phi]\cos 2\lambda\zeta,\ \exp[ik\phi]\sin 2\lambda\zeta,\ \exp[ik\phi],\ \exp[i2k\phi]$$

and their complex conjugates. Typically, the expansion of the mean motion $v_{00}(x, \tau)$ gives

$$\left.\begin{array}{l} A_c^2\colon\ (D^2 - 2a_0)F_1 = -\dfrac{1}{2\alpha}D(f_0 f_{20}), \\ |B_c|^2\colon\ D^2 - (b_0 + \tilde{b}_0)F_3 = -\dfrac{1}{2\alpha}D[h_{20}\tilde{h}_0 + \tilde{h}_{20}h_0], \end{array}\right\} \tag{3.17}$$

with

$$F_1 = F_3 = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}. \tag{3.18}$$

The corresponding equations for F_2 and F_4, together with the conditions (3.16), subject to the reasonable assumption that $2a_0$ and $(b_0 + \tilde{h}_0)$ are not eigenvalues of the homogeneous forms of equations (3.17), yield

$$F_2 = F_1, \quad F_4 = F_3. \tag{3.19}$$

† For $\mu = 0$, and $\lambda = \lambda_c$, $T = T_c$ the real part of $b_{c0} < 0$ for $k > 0$.

Once the first-order problems for a_0, b_0 and the functions f_0, f_{20}, h_0 and h_{20} have been solved, equations (3.17) can be integrated.

As one other example, consider the expansion for $v_{c22}(x,\tau)$, the coefficient of $\exp[i2k\phi]\cos 2\lambda\zeta$. The terms proportional to B_c^2 yield

$$\begin{aligned} M(2\lambda, 2b_0, 2k)p_1 = {} & (2\lambda^2/\alpha)D(h_{20}^2+h_{30}^2) - \tfrac{1}{2}\{(1/\alpha)N(2\lambda, 2b_0, 2k)(D^2-4\lambda^2)D \\ & + i2k(D^2+4\lambda^2)\}(h_{20}h_0) + (\lambda/\alpha)(D^2+4\lambda^2)(h_{20}h_{30}) - \{\lambda^2T + 2k^2\alpha D \\ & \qquad - ikN(2\lambda, 2b_0, 2k)(D^2-4\lambda^2)\}h_0^2 \\ & - \lambda\{(1/\alpha)N(2\lambda, 2b_0, 2k)(D^2-4\lambda^2) + i4kD\}(h_0h_{30}). \end{aligned} \tag{3.20}$$

The boundary conditions are

$$p_1 = D^2p_1 = N(2\lambda, 2b_0, 2k)Dp_1 = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}. \tag{3.21}$$

Again, once h_0, h_{20} and h_{30} have been determined there is no difficulty in integrating equation (3.20) for p_1, provided that $2b_0$ is not an eigenvalue of the corresponding homogeneous problem. This assumption is certainly valid for $\lambda = \lambda_c$, and T near T_c. Further, the terms proportional to B_s^2 in the expansion of v_{c22} and proportional to B_cB_s in the expansion of v_{s22} lead to equations similar to equations (3.20) and (3.21); from which it can be shown that

$$p_2 = -p_1, \quad q_1 = 2p_1. \tag{3.22}$$

Proceeding in the same manner, we find that the leading terms in the expansions for the mean motion and the first harmonics can be computed once the first-order eigenvalue and eigenfunction problems have been solved. In addition, upon using (3.16), we find the following relations:

$\underline{v_{c20},\ u_{c20},\ w_{s20},\ v_{s20},\ u_{s20},\ w_{c20}}$

$$\left.\begin{aligned} m_2 = -m_1, \quad m_{22} = -m_{21}, \quad m_{32} = -m_{31}; \quad m_4 = -m_3, \\ m_{24} = -m_{23}, \quad m_{34} = -m_{33}; \\ n_1 = 2m_1, \quad n_{21} = 2m_{21}, \quad n_{31} = -2m_{31}; \quad n_2 = n_3 = m_3, \\ n_{22} = n_{23} = m_{23}, \quad n_{32} = n_{33} = -m_{33}. \end{aligned}\right\} \tag{3.23}$$

$\underline{v_{c22},\ u_{c22},\ w_{s22},\ v_{s22},\ u_{s22},\ w_{c22}}$

$$\begin{aligned} p_2 = -p_1, \quad p_{22} = -p_{21}, \quad p_{32} = -p_{31}; \quad q_1 = 2p_1, \quad q_{21} = 2p_{21}, \\ q_{31} = -2p_{31}. \end{aligned} \tag{3.24}$$

$\underline{v_{c21},\ u_{c21},\ w_{s21},\ v_{s21},\ u_{s21},\ w_{c21}}$

$$\begin{aligned} r_2 = -r_1, \quad r_{22} = -r_{21}, \quad r_{32} = -r_{31}; \\ s_1 = s_2 = r_1, \quad s_{21} = s_{22} = r_{21}, \quad s_{31} = s_{32} = -r_{31}. \end{aligned} \tag{3.25}$$

$$\begin{array}{ll} \underline{v_{00}} & \underline{w_{00}} \\ F_2 = F_1, \quad F_4 = F_3. & \left.\begin{array}{l} G_3 = \tilde{G}_2, \quad G_1 = 0, \\ G_2 \text{ is purely imaginary.} \end{array}\right\} \end{array} \tag{3.26}$$

$\underline{v_{01},\ u_{01};\ v_{02},\ u_{02};\ w_{01},\ w_{02}}$

$$t_2 = t_1, \quad t_{22} = t_{21}; \quad y_2 = y_1, \quad y_{22} = y_{21}; \quad z_2 = -z_1; \quad z_3 = 0. \tag{3.27}$$

Third-order terms. Finally, we consider several typical third-order terms in the expansions of the fundamentals, v_{c10}, v_{s10}, v_{c11} and v_{s11}. The coefficient of A_c^3 in the expansion of $v_{c10}(x,\tau)$ yields

$$\begin{aligned} M(\lambda, 3a_0, 0)f_1 &= 2a_{c1}(D^2-\lambda^2)(D^2-\lambda^2-2a_0)f_0 \\ &+\left\{\frac{\lambda^2}{\alpha}D+\frac{\lambda}{2\alpha}(D^2+\lambda^2)\right\}(f_{20}m_{31}-f_{30}m_{21})-\tfrac{1}{2}\lambda^2T(2f_0F_1+f_0m_1) \\ &-(1/\alpha)N(\lambda,3a_0,0)(D^2-\lambda^2)[D(f_{20}F_1+\tfrac{1}{2}f_{20}m_1+\tfrac{1}{2}f_0m_{21}) \\ &\qquad +\lambda(f_{30}F_1+\tfrac{1}{2}f_0m_{31}-\tfrac{1}{2}f_{30}m_1)], \end{aligned} \tag{3.28}$$

with
$$f_1 = D^2f_1 = N(\lambda, 3a_0, 0)Df_1 - a_{c1}Df_0 = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}. \tag{3.29}$$

Except for the parameter a_{c1}, the non-homogeneous terms in equations (3.28) and (3.29) are known.† The determination of a_{c1} has been discussed elsewhere (Davey 1962; Reynolds & Potter 1967) and does not involve the non-axisymmetric components of the flow; the argument is the same as that for determining b_{c4} which is discussed later in this section. Once a_{c1} is determined, f_1 can be computed numerically.

Consider now the equation for $f_2(x)$ corresponding to the term $A_cA_s^2$ in the expansion for $v_{c10}(x,\tau)$. Making use of the relations (3.16) and (3.22)–(3.27), we find that the equation and boundary conditions for f_2 are identical with those for f_1 with (f_1, a_{c1}) replaced by (f_2, a_{c2}). Thus $a_{c2} = a_{c1}, f_2 = f_1$. A similar procedure for the remaining third-order terms in the expansion for v_{c10} and for the third-order terms in the expansion for v_{s10} yields the relations:

$$\left.\begin{aligned} &a_{s1} = a_{s2} = a_{c2} = a_{c1} = a_1, \quad g_1 = g_2 = f_2 = f_1; \\ &a_{s3} = a_{c3} = a_3, \quad g_3 = f_3; \\ &a_{s4} = a_{c4} = a_4, \quad g_4 = f_4; \\ &a_{s5} = a_{c5} = a_5, \quad a_{5r} = \tfrac{1}{2}(a_3-a_4), \quad g_5 = f_5, \quad f_{5r} = \tfrac{1}{2}(f_3-f_4); \\ &a_{s6} = a_{c6} = \tilde{a}_5, \quad g_6 = f_6 = \tilde{f}_5. \end{aligned}\right\} \tag{3.30}$$

Here r used as a subscript denotes the real part. The c and s subscripts are no longer necessary. Later we use i to denote the imaginary part.

Similarly, for the third-order terms in the expansions for v_{c11} and v_{s11}, we find

$$\left.\begin{aligned} &b_{s1} = b_{c1} = b_1, \quad l_1 = h_1; \\ &b_{s2} = b_{c2} = b_2, \quad l_2 = h_2; \\ &b_{s3} = b_{c3} = b_3, \quad l_3 = h_3; \\ &b_{s4} = b_{c4} = b_4, \quad l_4 = h_4; \\ &b_{s5} = b_{c5} = b_3 - b_4, \quad l_5 = h_5 = h_3 - h_4; \\ &b_{s6} = b_{c6} = b_1 - b_2, \quad l_6 = h_6 = h_1 - h_2. \end{aligned}\right\} \tag{3.31}$$

Again, the s and c subscripts may now be dropped.

† The term involving a_{c1} arises from the term $a_{c1}A_c^3$ in the expansion of $dA_c/d\tau$.

To illustrate how the parameters $a_1,\dots,a_5$, $b_1,\dots,b_4$ are determined, consider the equation for $h_4 = l_4$ which involves b_4, since this parameter, as well as a_1, plays an important role in the subsequent analysis. The equation for h_4 is

$$M(\lambda, b_0+2a_0, k)h_4 = 2b_4\{N(\lambda, b_0+a_0, k)(D^2-\lambda^2)+ik\alpha D\}h_0+\psi, \tag{3.32}$$

where

$$\left.\begin{aligned}\psi = {} & (\lambda^2/\alpha)D(f_{20}r_{21}-h_{20}m_{21}-f_{30}r_{31}-2f_{30}z_1+h_{30}m_{31}) \\ & -\tfrac{1}{2}\{(1/\alpha)N(\lambda,b_0+2a_0,k)(D^2-\lambda^2)D+ik(D^2+\lambda^2)\} \\ & \quad\times[f_{20}r_1+2h_{20}F_1-h_{20}m_1-h_0m_{21}+f_0r_{21}] \\ & -(\lambda/2\alpha)(D^2+\lambda^2)[-f_{20}r_{31}+2f_{20}z_1+h_{20}m_{31}-h_{30}m_{21}+f_{30}r_{21}] \\ & -\{\tfrac{1}{2}\lambda^2T+k^2\alpha D-ikN(\lambda,b_0+2a_0,k)(D^2-\lambda^2)\} \\ & \quad\times[2h_0F_1+f_0r_1-h_0m_1]+\tfrac{1}{2}\lambda\{(1/\alpha)N(\lambda,b_0+2a_0,k) \\ & \quad\times(D^2-\lambda^2)+i2kD\}[-2h_{30}F_1-f_0r_{31}+2f_0z_1+h_0m_{31}-h_{30}m_1 \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +f_{30}r_1].\end{aligned}\right\} \tag{3.33}$$

The boundary conditions are

$$h_4 = 0, \quad D^2h_4 = 0, \quad N(\lambda, b_0+2a_0, k)Dh_4 = b_4 Dh_0 \quad \text{at} \quad x = \pm\tfrac{1}{2}. \tag{3.34}$$

Again, except for the parameter b_4 the non-homogeneous terms in equations (3.32) and (3.34) are known.

To determine b_4 we proceed as follows. The homogeneous boundary-value problem corresponding to equations (3.32) and (3.34) is $M(\lambda, b_0+2a_0, k)h_4 = 0$ with the boundary conditions $h_4 = D^2h_4 = N(\lambda, b_0+2a_0, k)Dh_4 = 0$ at $x = \pm\frac{1}{2}$. But in the limit $T\to T_c$, $a_0 \to 0$, and we obtain precisely the same equation and boundary conditions as for the linear eigenvalue problem for b_0 and h_0, equations (3.13) and (3.15). Since this homogeneous boundary-value problem has a non-trivial solution (by the choice of b_0), we can anticipate a singular behaviour of the solution of the non-homogeneous boundary-value problem (3.32) and (3.34) as $T \to T_c$ unless b_4 is properly chosen. Indeed the solution will behave as a_0^{-1} as $a_0 \to 0$. To remove this singularity, it is necessary and sufficient† to set $a_0 = 0$ in (3.32)–(3.34) and to require that the integral from $-\frac{1}{2}$ to $\frac{1}{2}$ of h_0^+ times the right-hand side of (3.32) is equal to $b_4[Dh_0D^2h_0^+]_{-\frac{1}{2}}^{\frac{1}{2}}$. Here h_0^+, the adjoint of h_0, is the solution of the adjoint eigenvalue problem:

$$\left.\begin{aligned} & M(\lambda, b_0, k)h_0^+ = 0, \\ & h_0^+ = Dh_0^+ = (D^2-2\lambda^2-b_0-ik\Omega_l)D^2h_0^+ = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}.\end{aligned}\right\} \tag{3.35}$$

This procedure yields the condition

$$b_4 = \int_{-\frac{1}{2}}^{\frac{1}{2}} h_0^+(x)\psi\,dx \bigg/ \int_{-\frac{1}{2}}^{\frac{1}{2}} [h_0^+N(\lambda,b_0,k)(D^2-\lambda^2)h_0+h_0N(\lambda,b_0,k)(D^2-\lambda^2)h_0^+]dx, \tag{3.36}$$

where it is understood that the calculation is carried out at $a_0 = 0$.

The parameters a_1, a_3, a_4, a_5, b_1, b_2, b_3 and b_4 can be determined in a similar manner. The arguments of Watson (1960) have been applied by Davey (1962) to show that for a given T $(\neq T_c)$ within a supposed range of convergence of the

† See Ince (1956, §9.34).

expansions, there is no 'better' choice of a_1 than that given by the formula analogous to (3.36) when working to cubic order in amplitude; this is presumably true at higher order in amplitude. A corresponding statement applies to b_4 as given by (3.36).

4. The mathematical model

Using relations (3.16), (3.30) and (3.31), the amplitude equations (3.10) for A_c, A_s, B_c, and B_s to third order in amplitude take the form

$$\left.\begin{aligned} dA_c/d\tau &= a_0 A_c + a_1 A_c^3 + a_1 A_c A_s^2 + a_3 A_c |B_c|^2 + a_4 A_c |B_s|^2 \\ &\qquad + a_5 A_s B_c \tilde{B}_s + \tilde{a}_5 A_s \tilde{B}_c B_s, \\ dA_s/d\tau &= a_0 A_s + a_1 A_s^3 + a_1 A_s A_c^2 + a_3 A_s |B_s|^2 + a_4 A_s |B_c|^2 + a_5 A_c B_s \tilde{B}_c \\ &\qquad + \tilde{a}_5 A_c \tilde{B}_s B_c, \\ dB_c/d\tau &= b_0 B_c + b_1 B_c |B_c|^2 + b_2 B_c |B_s|^2 + b_3 B_c A_c^2 + b_4 B_c A_s^2 \\ &\qquad + (b_3 - b_4) B_s A_c A_s + (b_1 - b_2) \tilde{B}_c B_s^2, \\ dB_s/d\tau &= b_0 B_s + b_1 B_s |B_s|^2 + b_2 B_s |B_c|^2 + b_3 B_s A_s^2 + b_4 B_s A_c^2 \\ &\qquad + (b_3 - b_4) B_c A_s A_c + (b_1 - b_2) \tilde{B}_s B_c^2, \end{aligned}\right\} \quad (4.1)$$

where the a and b coefficients, which depend on the parameters of the problem, are determined by the procedure described in § 3.

We now discuss how the possible solutions of equations (4.1) and their stability characteristics as functions of the parameters λ, k, μ, and T provide a model for the transition with increasing T from Couette flow, through Taylor-vortex flow, to wavy-vortex flow. In particular we will consider the case $\mu = 0$. For assigned values of the wave-numbers λ and k, there is a critical value of T, $T_c(\lambda, k)$ such that for $T > T_c(\lambda, k)$ this particular disturbance grows according to linear theory. The minimum of $T_c(\lambda, k)$ over all positive λ and k determines T_c such that for $T < T_c$ all disturbances are damped, again according to linear theory. For $\mu = 0$, T_c occurs for $k = 0$ and $\lambda = \lambda_c$ as given in equations (3.1). For much of the discussion λ will be fixed equal to λ_c, but it will be necessary to consider the variation with k of $T_c(\lambda_c, k) > T_c(\lambda, 0) = T_c$. We shall normally use the notation $T_c(k)$, with especially $T_c(0) = T_c$. However, the notation $T_c(m)$, m being an integer, is also used. For the instability of the Taylor-vortex flow, according to non-linear theory, we denote the critical Taylor number, as a function of m, by $T'(m)$.

The calculations of Krueger *et al.* (1966) and Roberts (1965) indicate that, for values of k of physical interest, $T_c(k)$ increases monotonically with k, at least for the range of k of the calculations. Though the change in $T_c(k)$ is only a few per cent, it follows that for $\lambda = \lambda_c$ and an assigned value of $k \neq 0$, $a_0 > 0$ for $T > T_c$, but $b_{0r} < 0$ up to $T_c(k) > T_c$. This will be discussed in more detail in the next section.

The calculation of the parameters $a_1, \ldots, b_4$ has not been carried out in full except in a certain limiting case to be discussed in § 5. However, it is still useful to consider the general system (4.1). At the end of this section we will give some justification for truncating our expansions at third order, but first consider the significance of the possible equilibrium solutions of equations (4.1).

I(0)-*Laminar Couette flow.* A possible solution of equations (4.1) is

$$A_c = A_s = B_c = B_s = 0,$$

i.e. the periodic perturbation is zero. This solution, which gives the basic Couette flow, is stable to small perturbations in A_c, A_s, B_c and B_s provided that a_0 and $b_{0r} < 0$. This is the case for $T < T_c$, and Couette flow is stable within the model for $T < T_c$.

I(i). *Taylor-vortex flow.* It is easy to show that a second class of solutions of equations (4.1) is

$$B_c = B_s = 0; \quad A_s = CA_c, \quad A_c^2 + A_s^2 = Ka_0 e^{2a_0\tau}/(1 - Ka_1 e^{2a_0\tau}), \tag{4.2}$$

where K and C are arbitrary real constants. It is known that $a_1 < 0$ and $a_0 \gtrless 0$ as $T \gtrless T_c$. Hence for $T > T_c$,

$$(A_c^2 + A_s^2) \to (-a_0/a_1) = A_e^2 \quad \text{as} \quad \tau \to \infty. \tag{4.3}$$

This equilibrium solution represents a Taylor-vortex flow, the parameter C giving the ζ-phase. Without loss of generality C can be taken equal to zero. The azimuthal velocity perturbation from laminar Couette flow takes the form

$$\begin{aligned} v(x,\zeta,\tau) &= v_{00}(x,\tau) + v_{c10}(x,\tau)\cos\lambda\zeta + v_{c20}(x,\tau)\cos 2\lambda\zeta + \dots \\ &= [A_c^2 F_1 + O(A_c^4)] + [A_c f_0 + A_c^3 f_1 + O(A_c^5)]\cos\lambda\zeta \\ &\qquad + [A_c^2 m_1 + O(A_c^4)]\cos 2\lambda\zeta + \dots, \end{aligned} \tag{4.4}$$

and as $\tau \to \infty$ we obtain the equilibrium motion

$$v = A_e f_0(x)\cos\lambda\zeta + A_e^2[F_1(x) + m_1(x)\cos 2\lambda\zeta] + O(A_e^3). \tag{4.5}$$

For details of (4.4) and (4.5), the reader may consult the paper by Davey (1962), where the theoretically predicted torque and the experimentally measured torque are shown to be in excellent agreement for a small range of T above T_c.

Consider now small perturbations from the Taylor-vortex flow. We write

$$A_c(\tau) = A_e + \chi.$$

Linearization of (4.1) for small values of χ, A_s, B_c and B_s gives

$$\left.\begin{aligned} \frac{d\chi}{d\tau} = -2a_0\chi, \quad \frac{dA_s}{d\tau} = 0, \\ \frac{dB_c}{d\tau} = \left(b_0 - \frac{b_3}{a_1}a_0\right)B_c, \quad \frac{dB_s}{d\tau} = \left(b_0 - \frac{b_4}{a_1}a_0\right)B_s. \end{aligned}\right\} \tag{4.6}$$

Since $a_0 > 0$ for $T > T_c$ (where the Taylor-vortex flow exists), χ decays. The result for $dA_s/d\tau$ reflects the presence of the class of solutions (4.3) with an arbitrariness in the axial phase of the Taylor-vortex motion (see Segel (1965) for a similar situation in the thermal-convection problem). The real parts of $b_0 - b_3 a_0/a_1$ and $b_0 - b_4 a_0/a_1$ determine the growth of decay of B_c and B_s respectively, and hence the instability or stability of the Taylor-vortex motion with respect to in-phase and out-of-phase non-axisymmetric disturbances respectively. The calculation of the parameters a_0, a_1, b_3 and b_4 and the answer to these important questions will be discussed in § 5.

A natural question which might be raised is the relevance of the stability equations (4.6) to a 'straightforward' discussion of the stability of steady Taylor-vortex flow, the azimuthal component of which is given to second order in amplitude by (2.1) plus (4.5). The u and w components of velocity are given by formulae similar to (4.5). Suppose this Taylor-vortex flow is perturbed by a small non-axisymmetric disturbance of such a form that its azimuthal velocity is $v'(x,\zeta)\exp[q\tau+ik\phi]$. Upon expanding v' and q as series in the small amplitude $A_e = (-a_0/a_1)^{\frac{1}{2}}$, we can show that to second order in amplitude, $q = b_0 - b_3 a_0/a_1$ if v' has the same phase in ζ as the Taylor-vortex mode, but that $q = b_0 - b_4 a_0/a_1$ if v' differs in phase by $\frac{1}{2}\pi$. These formulae confirm the stability coefficients given by (4.6) through terms $O(A_e^2)$. In § 6 we shall discuss the possible influence of terms $O(A_e^4)$ on the precise location of the zeros of the stability coefficients (4.6), that is of q above.

Other solutions of equations (4.1) are tabulated below. Stability criteria are given for those cases where the result can be stated simply; this is the case for the 'simple' modes, which have only one of A_c, A_s, B_c, B_s non-zero, and for the spiral mode $B_s = \pm iB_c$. In these solutions τ_s denotes an arbitrary time phase.

I(ii). *Non-axisymmetric simple mode.*

$$\left.\begin{aligned} &A_c = A_s = B_s = 0, \quad B_c = \beta_e e^{i\omega(\tau-\tau_s)}, \\ &\beta_e = (-b_{0r}/b_{1r})^{\frac{1}{2}}, \qquad \omega = b_{0i} - b_{1i}b_{0r}/b_{1r}. \end{aligned}\right\} \tag{4.7}$$

This solution exists if $-b_{0r}/b_{1r} > 0$, and it is stable if $a_0 - a_3 b_{0r}/b_{1r}$, $a_0 - a_4 b_{0r}/b_{1r}$, $-b_{0r}$ and $b_{0r}(1-b_{2r}/b_{1r})$ are all less than zero. A generalization of this motion is $A_c = A_s = 0$, $B_c = \beta_{ce}\exp[i\omega(\tau-\tau_s)]$, $B_s = \beta_{se}\exp[i\omega(\tau-\tau_s)]$ with $\beta_{ce}^2+\beta_{se}^2 = \beta_e^2$.

II(i). *Wavy-vortex flow.*

$$\left.\begin{aligned} &A_s = B_c = 0, \quad A_c^2 = A_e^2 = \frac{a_0 b_{1r} - a_4 b_{0r}}{a_4 b_{4r} - a_1 b_{1r}}, \quad B_s = \beta_e e^{i\omega(\tau-\tau_s)}, \\ &\beta_e^2 = \frac{a_1 b_{0r} - a_0 b_{4r}}{a_4 b_{4r} - a_1 b_{1r}}, \quad \omega = b_{0i} + b_{1i}\beta_e^2 + b_{4i}A_e^2. \end{aligned}\right\} \tag{4.8}$$

This solution, which represents the interaction of a Taylor-vortex mode with an out-of-phase non-axisymmetric mode, is of particular interest. It exists when $a_1 b_{0r} - a_0 b_{4r}$, $a_4 b_{4r} - a_1 b_{1r}$ and $a_0 b_{1r} - a_4 b_{1r}$ all have the same sign. If only A_c and B_s modes are allowed, it is stable if $a_1 A_e^2 + b_{1r}\beta_e^2 < 0$ and $a_1 b_{1r} - a_4 b_{4r} > 0$. Note that if the latter condition holds, then the existence of the wavy-vortex flow requires $a_1 b_{0r} - a_0 b_{4r} < 0$, which in turn implies that the Taylor-vortex flow I (i) (with $a_1 < 0$, $a_0 > 0$) is unstable according to (4.6) to a B_s perturbation! For A_s and B_c perturbations the corresponding statement is much more complex. The form of the azimuthal component of the disturbance velocity (3.2), correct through terms $O(A_e^2)$ and $O(\beta_e^2)$, is

$$\begin{aligned} v(x,\tau,\phi,\zeta) = {} & A_e f_0(x)\cos\lambda\zeta + 2\{\beta_e h_0(x)\sin\lambda\zeta\exp(i[k\phi+\omega(\tau-\tau_s)])\}_r \\ & + \{A_e^2 F_1(x) + \beta_e^2 F_3(x)\} + \{A_e^2 m_1(x) + \beta_e^2 m_3(x)\}\cos 2\lambda\zeta \\ & + 2\beta_e^2\{[-p_1(x)\cos 2\lambda\zeta + y_1(x)]\exp(i2[k\phi+\omega(\tau-\tau_s)])\}_r \\ & + 2A_e\beta_e\{r_1(x)\sin 2\lambda\zeta\exp(i[k\phi+\omega(\tau-\tau_s)])\}_r. \end{aligned} \tag{4.9}$$

This result should be compared with the Taylor-vortex flow (4.5).

II(ii). *Non-axisymmetric vortex flow.*

$$\left.\begin{aligned} &A_s = 0, \quad B_s = 0, \quad A_c = A_e = \frac{a_0 b_{1r} - a_3 b_{0r}}{a_3 b_{3r} - a_1 b_{1r}}, \quad B_c = \beta_e \exp\{i\omega(\tau - \tau_s)\}, \\ &\beta_e^2 = \frac{a_1 b_{0r} - a_0 b_{3r}}{a_3 b_{3r} - a_1 b_{1r}}, \quad \omega = b_{0i} + b_{1i}\beta_e^2 + b_{3i} A_e^2. \end{aligned}\right\} \quad (4.10)$$

This solution exists if the numerators and denominators of A_e^2 and β_e^2 respectively are of the same sign. We do not consider the question of stability here. The terminology 'non-axisymmetric vortex flow' is used to indicate that while the motion depends on ϕ, the A_c and B_c disturbances are in phase in ζ, and the vortex motion is bounded by planes of constant ζ.

II(iii). *Spiral mode.*

$$\left.\begin{aligned} A_c = A_s = 0, \quad B_c = \beta_e \exp[i\omega(\tau - \tau_s)], \quad B_s = \beta_e \exp[i\omega(\tau - \tau_s) \pm \tfrac{1}{2} i\pi], \\ \beta_e^2 = -b_{0r}/2b_{2r}, \quad \omega = b_{0i} + 2b_{2i}\beta_e^2. \end{aligned}\right\} \quad (4.11)$$

This solution exists if $-b_{0r}/2b_{2r} > 0$. It is stable if $b_{0r} > 0$, $b_{1r}/b_{2r} > 1$, and $a_0 - (a_3 + a_4)b_{0r}/2b_{2r} < 0$. Because of the phase difference of $\frac{1}{2}\pi$ in B_c and B_s the terms in the velocity distribution combine together to give a wave travelling in both the ζ and ϕ directions. Alternatively, it may be regarded as a spiral pattern which rotates with a certain angular velocity.

III *and* IV. Unless certain relations exist between the a and b coefficients, there are no triple modes. Quadruple modes are either generalizations of the double modes A_c, B_s and A_c, B_c or (in principle) completely new modes. The latter possibility has not been investigated.

In concluding this section we return to the questions raised earlier of truncating the expansions at third order in the amplitudes. Assuming that a_1, a_3, a_4, a_5, b_1, b_2, b_3 and b_4 are $O(1)$, we see from equations (4.3), that possible equilibrium solutions of (4.1) have amplitudes which are $O(a_0^{\frac{1}{2}})$ and/or $O(b_{0r}^{\frac{1}{2}})$. For a given value of the Taylor number b_{0r}/a_0 is fixed, so that the amplitudes are $O(a_0^{\frac{1}{2}})$ where a_0 is a small number if T is close to T_c. Whereas the linear and cubic terms on the right-hand side of equations (4.1) are then both $O(a_0^{\frac{3}{2}})$, the terms omitted which are of quintic or higher order in amplitude are $O(a_0^{\frac{5}{2}})$.

To give some justification for neglecting such terms, we will discuss global stability of the system (4.1) in order to assess whether all solutions of amplitude $O(a_0^{\frac{1}{2}})$ retain that order, or whether there are solutions which become unbounded as $\tau \to \infty$. If any solutions had such behaviour, it would be invalid to truncate the set of ordinary differential equations (4.1) at cubic terms. Happily, however, it is possible to obtain conditions on the coefficients a_1, b_1, etc., which ensure that solutions within some bounded domain $O(a_0^{\frac{1}{2}})$ cannot escape from that domain; so that, consequently, solutions of equations (4.1) retain amplitudes $O(a_0^{\frac{1}{2}})$. This statement of global stability can be proved by constructing a Liapunov function (Minorsky 1962) of the form

$$L = \alpha_c A_c^2 + \alpha_s A_s^2 + \beta_c |B_c|^2 + \beta_s |B_s|^2, \quad (4.12)$$

where α_c, α_s, β_c, β_s are real and positive. It is found that $dL/d\tau$ is negative if L is greater than some value of order a_0, provided certain conditions hold on a_1, b_1, etc. This result ensures that, for amplitudes larger than $O(a_0^{\frac{1}{2}})$ the trajectories are heading 'inwards'; consequently, the system (4.1) is globally stable. The conditions on a_1, b_1, etc., are satisfied (Stuart 1964) for the 'simplified' mathematical model proposed in the next section. We note, in passing, that if the cubic terms in (4.1) were ignored (as in linear theory), the system would not be globally stable when $a_0 > 0$ and/or $b_{0r} > 0$, since then there would be exponentially increasing solutions.

Before leaving the topic of the truncation of the amplitude equations at cubic terms, we reiterate a second point mentioned earlier in this section. At values of the parameters near those at which the 'local' stability of an equilibrium solution changes character, the quintic terms may be important in a specification of the *precise location* of that change in character. This will be discussed in detail in §6.

5. A simplified mathematical model

In order to complete our discussion of the possible equilibrium solutions of equations (4.1) and their stability, we need to compute numerical values for the coefficients $a_0, a_1, \ldots, a_5, b_0, \ldots, b_4$. Of particular interest are the combinations

$$b_{0r} - \frac{b_{3r}}{a_1} a_0 \quad \text{and} \quad b_{0r} - \frac{b_{4r}}{a_1} a_0, \tag{5.1}$$

which, as indicated by (4.6), determine the stability of the Taylor-vortex flow. Since $a_0 = 0$ at $T = T_c(0)$, each equals b_{0r} and hence is negative. Assuming, however, that our theory is a good model of the physical problem, we may anticipate that for a fixed value of λ ($=\lambda_c$) and within a range of $T > T_c$ there will be a minimum value of T, say T', greater than T_c with a corresponding critical value of k for which one of (5.1) will become positive. Above that value of T the Taylor-vortex motion will be unstable, and the corresponding value of k will determine the azimuthal wave number† of the critical non-axisymmetric disturbance.

A calculation of the a's and b's as functions of k and T for fixed $\mu(=0)$ and $\lambda(=\lambda_c)$ is a formidable task, and it is natural to consider possible simplifications. One such simplification is suggested by the work of Gross (1964), as reported in part by Krueger *et al.* (1966) which was mentioned earlier. Gross showed that, for $\mu = 0$ and a given value of λ, both the critical Taylor numbers and the amplification rates depend only slightly on k, even for (quite typical) values of k of order 10. For the case of small-gap, his results are displayed in table 1. Note that while $T_c(m)$ is monotonic increasing, the amplification rate as indicated by (db_{0r}/dT) evaluated at $T_c(m)$ is monotonic decreasing. The actual critical values of λ and T for the assigned values of κ are given in the last two columns of table 1; they show that there are only slight changes if λ_c is replaced by $\lambda_c(m)$. Gross showed further that while results from the small-gap equations are changed by a few

† We recall (from below (3.21)), that although $k = m\Omega_0 d^2/\nu$ is a continuous parameter, it is only necessary to consider those values which correspond to integers, m, for the physical wave-number. From the definitions of T and α in (2·5) with $d/R_0 \to 0$ in T, it follows that for $\mu = 0$, $k = m(T\delta)^{\frac{1}{2}}/2 = \kappa(\frac{1}{2}T)^{\frac{1}{2}}$ where $\kappa = m(\frac{1}{2}\delta)^{\frac{1}{2}}$ and $\delta = d/R_0$; thus for a given geometry, values of $\kappa = m(\frac{1}{2}\delta)^{\frac{1}{2}}$ are assigned.

percent if the exact linearized equations are used, *these errors occur uniformly.* For example, using the exact equations with $\mu = 0$, $\eta = 0{\cdot}95$ (i.e. $\delta \simeq 0{\cdot}05$), $\lambda = 3{\cdot}127$, he found that $T_c = 1755$, $\alpha = 0{\cdot}00746$ for $m = 0$ and $T_c = 1763$, $\alpha = 0{\cdot}00742$ for $m = 1$. A comparison of these results with those in the first two lines of table 1 shows that the correction for gap size is essentially the same for $m = 1$ as for $m = 0$; it may also be helpful to refer to figure 4 on p. 535 of Krueger *et al.* (1966).

$\kappa = m(\frac{1}{2}\delta)^{\frac{1}{2}}$	m	$T_c(m)$ for $\lambda = 3{\cdot}127$	$\alpha(m) = (db_{0r}/dT)T_c(m)$† for $\lambda = 3{\cdot}127$	$\lambda_c(m)$	$T_c(m)$ for $\lambda = \lambda_c(m)$
0	0	1695	0·00772	3·127	1695
0·15811	1	1701	0·00770	3·131	1701
0·31623	2	1720	0·00758	3·143	1720
0·47434	3	1753	0·00740	3·163	1752
0·63246	4	1800	0·00720	3·190	1799
0·79057	5	1866	0·00690	3·225	1863

† The values of db_{0r}/dT at $T_c(m)$ have been inferred from Gross's data (1964, table 9).

TABLE 1. Data from linearized theory ($\delta = 0{\cdot}05$, $\mu = 0$)

Assuming for the moment that the coefficients of the non-linear terms in equations (4.1) are also only slightly dependent upon k (or κ), we consider the form taken by the a's and b's as $k \to 0$. Even for the case $m = 4$, $\delta = \frac{1}{20}$, for which the change in $T_c(m)$ from $T_c(0)$ is about 6 %, it is hoped that results in this limit will at least give qualitative information.

For the first-order terms, from (3.11) to (3.14),

$$b_0 = a_0; \quad h_0 = f_0, \quad h_{20} = f_{20}, \quad h_{30} = f_{30} \tag{5.2}$$

in the limit $k \to 0$. Using these results in the second-order equations, as typified by (3.17) and (3.20), and letting $k \to 0$, we find

$$\left.\begin{aligned} F_3 = 2F_1, \quad G_2 = 0;\\ m_3 = 2m_1, \quad m_{23} = 2m_{21}, \quad m_{33} = 2m_{31};\\ p_1 = m_1, \quad p_{21} = m_{21}, \quad p_{31} = m_{31};\\ r_1 = 2m_1, \quad r_{21} = 2m_{21}, \quad r_{31} = 2m_{31};\\ y_{21} = 0, \quad y_1 = F_1; \quad t_{21} = 0, \quad t_1 = 2F_1. \end{aligned}\right\} \tag{5.3}$$

Finally, from the third-order equations, as typified by (3.28), the same limit gives

$$\left.\begin{aligned} a_3 = 6a_1, \quad f_3 = 6f_1; \quad a_4 = 2a_1, \quad f_4 = 2f_1; \quad a_5 = 2a_1, \quad f_5 = 2f_1;\\ b_1 = 3a_1, \quad h_1 = 3f_1; \quad b_2 = 2a_1, \quad h_2 = 2f_1;\\ b_3 = 3a_1, \quad h_3 = 3f_1; \quad b_4 = a_1, \quad h_4 = f_1. \end{aligned}\right\} \tag{5.4}$$

The implication of these results for the stability of Taylor-vortex flow is

$$b_{0r} - \frac{b_{3r}}{a_1}a_0 = -2a_0, \quad b_{0r} - \frac{b_{4r}}{a_1}a_0 = 0 \quad \text{as} \quad k \to 0. \tag{5.5}$$

Thus for a discussion of the stability of the Taylor-vortex mode (A_c) with respect to in-phase (B_c) non-axisymmetric disturbances, it is probably valid to set $k = 0$. Moreover, since $a_0 > 0$ for $T > T_c$ the Taylor-vortex mode is stable to in-phase non-axisymmetric disturbances. On the other hand (4.6) and (5.5) show that stability with respect to out-of-phase (B_s) non-axisymmetric disturbances depends upon the correction terms in k, precisely because $b_0 \to a_0$ and $b_4 \to a_1$ as $k \to 0$. To examine the behaviour of $[b_{0r} - (b_{4r}/a_1)a_0]$ more carefully as $k \to 0$ with $\lambda = \lambda_c$ and $T - T_c$ small, we note that it can be shown that (with $m = 2k(T\delta)^{-\frac{1}{2}}$)

$$a_0(T) = \alpha(0)[T - T_c(0)] + O[T - T_c(0)]^2, \tag{5.6}$$

and

$$T_c(m) = T_c(0) + k^2 T_{c1} + \dots, \tag{5.7}$$

$$\begin{aligned} b_{0r}(m, T) &= \alpha(m)\,[T - T_c(m)] + O[T - T_c(m)]^2 \\ &= [\alpha(0) + k^2\alpha_1 + \dots]\{T - [T_c(0) + k^2 T_{c1} + \dots]\} + O[T - T_c(m)]^2 \\ &= \alpha(0)\,[T - T_c(0)] + k^2\{\alpha_1[T - T_c(0)] - \alpha(0)T_{c1}\} + O(k^4) + O[T - T_c(m)]^2, \end{aligned} \tag{5.8}$$

$$b_{4r}(m) = a_1 + k^2 b_{4r1} + O(k^4).\dagger \tag{5.9}$$

Neglecting terms $O[T - T_c(m)]^2$, $O[T - T_c(0)]^2$ and $O(k^4)$,

$$b_{0r} - \frac{b_{4r}}{a_1} a_0 = k^2 \left\{ \left[\alpha_1 - \frac{b_{41r}}{a_1} \alpha(0) \right] [T - T_c(0)] - \alpha(0) T_{c1} \right\}. \tag{5.10}$$

Thus, the stability or instability of the Taylor-vortex flow to an out-of-phase non-axisymmetric disturbance depends to first-order upon the k^2 correction factor (5.10).

At $T = T_c(0)$ it is clear that, since $\alpha(0) > 0$ and $T_{c1} > 0$, (5.10) must be negative. If, at some critical value of T, (5.10) changes sign, the Taylor-vortex flow will be unstable for higher values of T provided the terms neglected in (5.10) are truly unimportant. Moreover, since the azimuthal wave-number m appears only through k, any critical value of T derived from (5.10) must be independent of m; terms $O(k^4)$ would have to be added to (5.10) to obtain such a dependence on m. We shall return to this matter in our discussion of the results in § 6.

The preceding argument suggests that a simplified model may be considered with a_3, a_4, a_5, b_1, b_2 and b_3 of (4.1) replaced by their limiting values as $k \to 0$ as given in equations (5.4), but with $a_0(T)$, $b_0(m, T)$ and $b_4(m)$ taking exact values (for given μ, λ). As it affects the question of *instability* of Taylor-vortex flow with respect to B_s disturbances, the simplified model has the same accuracy as (4.1), since the B_s equation of (4.6) is exact; but its accuracy is less in the determination of the equilibrium state arising *from* that instability. In our study of the simplified model, we shall find it convenient to write

$$(X, Y, Z, V) = (-a_1)^{\frac{1}{2}}(A_c, A_s, B_c, B_s), \tag{5.11}$$

$$(\epsilon, \sigma, \gamma a_1) = (a_0, b_0, b_4), \tag{5.12}$$

† As discussed at the end of § 3, a_1 and b_{4r} are evaluated at T_c.

where γ, which is a complex number, approaches unity as $k \to 0$. Then equations (4.1) become

$$\left.\begin{aligned}
dX/d\tau &= \epsilon X - X^3 - XY^2 - 6X|Z|^2 - 2X|V|^2 - 2Y(Z\tilde{V} + \tilde{Z}V),\\
dY/d\tau &= \epsilon Y - Y^3 - YX^2 - 6Y|V|^2 - 2Y|Z|^2 - 2X(Z\tilde{V} + \tilde{Z}V),\\
dZ/d\tau &= \sigma Z - 3Z|Z|^2 - 2Z|V|^2 - 3ZX^2 - \gamma ZY^2 - (3-\gamma)VXY - \tilde{Z}V^2,\\
dV/d\tau &= \sigma V - 3V|V|^2 - 2V|Z|^2 - 3VY^2 - \gamma VX^2 - (3-\gamma)ZXY - \tilde{V}Z^2.
\end{aligned}\right\} \tag{5.13}$$

Corresponding to the possible equilibrium solutions of equations (4.1), we have the following possible equilibrium solutions of the simplified system (5.13). In stating the solutions, the facts that $\epsilon > \sigma_r$ and $a_1 < 0$ have been used, and a suffix e denotes the equilibrium value. Note that generalizations of the solutions can be effected by altering the ζ and τ phases, as was pointed out in the previous section.

I(0). *Laminar Couette flow.*

$$X \equiv Y \equiv Z \equiv V = 0 \tag{5.14}$$

exists for all ϵ and is stable for $\epsilon < 0$ ($T < T_c$), and unstable for $\epsilon > 0$ ($T > T_c$).

I(i). *Taylor-vortex flow.*

$$X_e^2 = \epsilon, \quad Y \equiv Z \equiv V \equiv 0 \tag{5.15}$$

exists for $\epsilon > 0$ ($T > T_c$) and is stable or unstable as $\sigma_r \lessgtr \gamma_r \epsilon$.

I(ii). *Non-axisymmetric simple mode.*

$$Z_e = (\tfrac{1}{3}\sigma_r)^{\frac{1}{2}} \exp(i\sigma_i \tau), \quad (X \equiv Y \equiv V \equiv 0) \tag{5.16}$$

exists for $\sigma_r > 0$ ($T > T_c(m)$), but is unstable.

II(i). *Wavy-vortex flow.*

$$X_e^2 = \frac{3\epsilon - 2\sigma_r}{3 - 2\gamma_r}, \quad V_e = \left(\frac{\sigma_r - \gamma_r \epsilon}{3 - 2\gamma_r}\right)^{\frac{1}{2}} e^{i\omega\tau}, \quad (Y \equiv Z \equiv 0), \tag{5.17}$$

where $\omega = \sigma_i - \gamma_i^2 X_e^2$. It exists if $\gamma_r < 1$ and $\sigma_r > \gamma_r \epsilon$. Whenever it exists, it is stable. Note that if $\sigma_r > \gamma_r \epsilon$, the Taylor-vortex flow is unstable.

II(ii). *Non-axisymmetric vortex flow.*

$$X_e^2 = \frac{2\sigma_r - \epsilon}{5}, \quad Z_e = \left(\frac{\epsilon - \sigma_r/3}{5}\right)^{\frac{1}{2}} \exp(i\sigma_i \tau), \quad (Y \equiv V \equiv 0) \tag{5.18}$$

exists for $\frac{1}{2}\epsilon < \sigma_r < 3\epsilon$, which is expected for some $T > T_c(m)$, but is unstable.

II(iii). *Spiral mode.*

$$Z_e = \tfrac{1}{2}\sigma_r^{\frac{1}{2}} \exp(i\sigma_i \tau), \quad V_e = \pm iZ_e, \quad (X \equiv Y \equiv 0) \tag{5.19}$$

exists for $\sigma_r > 0$ ($T > T_c(m)$), and is stable or unstable as $\sigma_r \gtrless \frac{1}{2}\epsilon$.

There are no triple modes since $\gamma_i \neq 0$, as we shall shortly see, and the quadruple modes are simply generalizations of the double modes II(i) and II(ii).

The most important observations (assuming there exists a $T > T_c$ such that $\sigma_r = \gamma_r \epsilon$) are:

(i) $\epsilon < 0$. Laminar Couette flow exists and is stable.

(ii) $\epsilon > 0$, $\sigma_r < \gamma_r \epsilon$. Laminar Couette flow is unstable, and Taylor vortex flow exists and is stable.

(iii) $\sigma_r > \gamma_r \epsilon$. Taylor-vortex flow is unstable, and wavy-vortex flow exists and is stable.

Perhaps of lesser, though notable, importance is the existence and stability of the spiral mode II (iii) for $\sigma_r > \frac{1}{2}\epsilon$.

Turning now to the computation of ϵ, σ and γ (or, equivalently, a_0, b_0, a_1 and b_4) we note, as mentioned earlier, that a_0 and a_1 have been computed by Davey (1962) and a_0 and b_0 by Gross (1964). It remains only to compute b_4. However, since, as part of the calculation of b_4, it is necessary to recompute a_0 and b_0 and since, moreover, $b_4 = a_1$ at $k = 0$, there is no practical advantage in our using the earlier results. Thus for given values of μ, λ, k and T we first solve (numerically)

m	b_{4r}	b_{4i}	b_{4r}/a_1
0	−10·035	0	1
1	−9·5063	0·72035	0·94731
2	−7·9569	1·5238	0·79291
4	−2·2664	3·6251	0·22585

TABLE 2. Non-linear coefficients ($\delta = 0{\cdot}05$, $\mu = 0$, $\lambda = 3{\cdot}12657$, $T = 1694{\cdot}95$). Scaling is $f_0(0) = 1$; h_0 scaling is irrelevant for b_4, a_1.

the linear stability problems for axisymmetric and non-axisymmetric disturbances to determine $a_0, f_0, f_{20}, f_{30}, f_0^+$ and b_0, h_0, h_{20}, h_{30}, h_0^+ respectively, where f_0^+ and h_0^+ are the adjoints of f_0 and h_0 respectively; then the second-order terms $m_1, m_{21}, m_{31}, r_1, r_{21}, r_{31}, z_1$ and F_1 are evaluated. Finally, b_4 and a_1 (by setting $k = 0$) are determined by the condition (3.36). The computational methods and procedure are broadly similar to those described in an earlier paper on Taylor-vortex flows (Davey 1962). Here, however, there is the added complexity brought about by the complex arithmetic and the vast number of functions and their derivatives which are involved, as a glance at (3.32) and (3.33) will show.

The critical Taylor number, $T_c = T_c(0)$, given by the calculations is

$$T_c(0) = 1694{\cdot}95 \quad \text{at} \quad \lambda = 3{\cdot}12657 \quad \text{with} \quad \mu = 0, \quad k = 0. \tag{5.20}$$

For $\mu = 0$ and the value of λ given in (5.20), a_1 and $b_4(m)$ have been evaluated at T_c, as discussed in §3, and for integer values of m for $\delta = 0{\cdot}05$ with $k = m(T\delta)^{\frac{1}{2}}/2$. The results are given in table 2.

The amplification rates and frequencies, given by the real (b_{0r}) and imaginary (b_{0i}) parts of b_0, are given in table 3 for several values of $m(\delta = \frac{1}{20})$ and a range of values of T, but with λ and μ given by (5.20).

The stability coefficient $b_{0r} - b_{4r}a_0/a_1 = \sigma_r - \gamma_r \epsilon$ as a function of T is shown in table 4. It can be seen that, for each value of m, the stability coefficient changes sign at some $T'(m)$, the value being about 1821 for $m = 1$, 1824 for $m = 2$, and 1834 for $m = 4$. The meaning of these results, the effects of the approximations, and the relevance of the theory to experiment, are discussed in §6. For the moment we note that the values of $b_{0r} - b_{4r} a_0/a_1$ for $m = 2$ and $m = 4$ are, to a rough approximation, 4 times and 16 times the values for $m = 1$ respectively.

m ...	0	1		2		4	
T	a_0	b_{0r}	$-b_{0i}$	b_{0r}	$-b_{0i}$	b_{0r}	$-b_{0i}$
1694·95	0·00000	−0·04791	4·8441	−0·19170	9·6901	−0·76789	19·396
1715	0·15444	+0·10611	4·8738	−0·03895	9·7495	−0·62023	19·515
1735	0·30771	0·25896	4·9032	+0·11263	9·8084	−0·47371	19·633
1755	0·46021	0·41104	4·9325	0·26345	9·8670	−0·32794	19·750
1775	0·61195	0·56235	4·9616	0·41351	9·9252	−0·18292	19·867
1795	0·76293	0·71292	4·9906	0·56282	9·9832	−0·03862	19·983
1815	0·91317	0·86275	5·0194	0·71140	10·041	+0·10496	20·099
1835	1·0627	1·0119	5·0480	0·85926	10·098	0·24784	20·214
1855	1·2115	1·1602	5·0765	1·0064	10·155	0·39001	20·328
1865	1·2856	1·2342	5·0907	1·0797	10·184	0·46084	20·385

TABLE 3. Amplification rates of linear theory ($\delta = 0{\cdot}05$, $\mu = 0$, $\lambda = 3{\cdot}12657$)

m ... T	1	2	4
1694·95	−0·04791	−0·19170	−0·76789
1715	−0·04020	−0·16141	−0·65511
1735	−0·03254	−0·13136	−0·54320
1755	−0·02492	−0·10146	−0·43188
1775	−0·01735	−0·07171	−0·32113
1795	−0·00981	−0·04211	−0·21093
1815	−0·00231	−0·01266	−0·10128
1835	+0·00515	+0·01664	+0·00783
1855	0·01258	0·04581	0·11640
1865	0·01629	0·06034	0·17048

TABLE 4. The stability coefficient $b_{0r} - b_{4r} a_0/a_1$ ($\delta = 0{\cdot}05$, $\mu = 0$, $\lambda = 3{\cdot}12657$)

This is in accordance with equation (5.10). Indeed the data in table 4 can be used to justify the following approximate formulae, valid for $\delta = d/R_0 = \frac{1}{20}$:

$$b_{0r} - b_{4r} a_0/a_1 = k^2\{1{\cdot}8264 \times 10^{-5}[T - T_c(0)] - 0{\cdot}22848 \times 10^{-2}\} - (0{\cdot}086202 \times 10^{-5})k^4, \tag{5.21}$$

where $k^2 = 21{\cdot}1875 m^2$, and

$$T'(m) = T_c(0) + 125 + m^2. \tag{5.22}$$

The k^4 effect, which leads to m^2 in (5.22), is seen to be relatively small. While the values of $T'(m)$ for $m = 1, 2, 4$ are close together, they are quite separate from $T_c(0)$.

6. Discussion of theory and comparison with experiment

It has been argued in the previous section, and confirmed by calculation, that instability of the Taylor-vortex flow is connected with subtle changes in the stability coefficient $(b_{0r} - b_{4r} a_0/a_1)$. Indeed, we have found that these changes do produce a reversal of sign of the above coefficient at some value of T of order 8 % above the Taylor number at which the Taylor-vortex flow itself appears. It may be concluded, therefore, that if the approximations we have made are valid, the mathematical model does yield an instability of the Taylor-vortex flows; later in this paper we shall compare the consequences of this theoretically derived instability with experiment. In the meantime, however, it is desirable to assess the magnitude of known approximations in the mathematical model, especially insofar as they affect the precise value of the Taylor number at which the coefficient $(b_{0r} - b_{4r} a_0/a_1)$ changes sign.

The approximations are of four main kinds: (i) the amplification rates $a_0(=\epsilon)$ and b_{0r} $(=\sigma_r)$ are regarded as small enough for the amplitude equations (4.1) to be truncated at cubic terms; (ii) the small-gap approximation $(d/R_0 \rightarrow 0)$ has been used both in the derivation of the basic partial differential equations (2.4) and, by implication, in the limit $k \rightarrow 0$ used in much of the analysis of the simplified mathematical model of §5; (iii) the number of basic modes has been restricted to four, all of which have the same *axial* wave-number whereas two of them have an *azimuthal* wave-number of zero and two have some other given azimuthal wave-number; (iv) the cylinders are infinitely long.

Let us consider now the question of possible instabilities of the Taylor-vortex flow (as opposed to the question of the nature of the motion which develops *from* any instabilities). Clearly, in the present work, attention is restricted to instabilities associated with perturbations of the form of the four basic modes, whose wave-numbers are subject to the restrictions described under (iii) above. Although it would be of value to have information concerning the stability or instability of Taylor-vortex flow against other perturbations, the four modes were originally selected because of their especial relevance to several experiments. In this sense the choice is felt to have been a sensible one. Nothing more can be said about the more complete problem at this stage, but the generalization from (iii) needs to be borne in mind.

The small-gap approximation (ii) is known to be a good approximation in linearized theory (Krueger *et al.* 1966), and further the error is uniform in k. The accuracy of the approximation for the non-linear problem cannot, however, be assessed until calculations have been done from the full equations (without $d/R_0 \rightarrow 0$). It is hoped that, as in linearized theory, the results are accurate to within 5 % or so (and if uniform in k would not change the critical azimuthal wave-number), at least with reference to the instability arising from the B_s mode. The additional approximation $k \rightarrow 0$ is also a good approximation in linearized theory, and has some validity for the non-linear problem in the sense that for $m = 1$, which corresponds to the smallest value of k used, the value of b_{4r}/a_1 differs by only about 5 % from unity (see table 3). Other aspects of these approximations, especially a reconsideration of the result that B_c perturbations decay, will be discussed later in this section.

We now come to what is, in principle, a major approximation in the calculation of the critical value, $T'(m)$, for instability of the Taylor-vortex flow; this approximation, namely (i) above, has already been hinted at in §4, but we must now discuss the matter with some care. Clearly, in obtaining this new critical condition, we have balanced out the two parts of $(b_{0r} - b_{4r} a_0/a_1)$, each of those parts being proportional to a 'small' parameter. (Although, as seen in table 3, a_0 and b_{0r} are of order 1 in the range of interest, this value is small compared with $\lambda^2 \sim 10$; the case for this criterion of 'smallness' has been argued elsewhere by Davey (1962, p. 346).) The neglected terms, due to truncation of the amplitude equations (4.1) at cubic terms, may well be smaller than each of the terms b_{0r} and $b_{4r}a_0/a_1$ individually, but may be very important in the neighbourhood of the zero of that coefficient. Indeed, it is conceivable that such terms could even prevent the occurrence of a zero in the range of validity of the present type of theory. It is for this reason that we must give detailed attention to this question.

The first and second parts of $(b_{0r} - b_{4r} a_0/a_1)$ arise respectively from linear and non-linear (cubic) terms of the relevant equations of (4.1). Let us consider the form which those equations would take if quintic terms were included. But rather than deal with the complete set (4.1), we restrict our attention to the pair of equations for the Taylor-vortex mode (A_c) subject to *linearized* B_s perturbations. It is this procedure that led to equations (4.6) at cubic order, and therefore to the stability coefficient $(b_{0r} - b_{4r} a_0/a_1)$. If quintic terms are included, (4.1) in a generalized form yields

$$\frac{dA_c}{d\tau} = a_0 A_c + a_1 A_c^3 + a_2 A_c^5 + \dots, \tag{6.1}$$

$$\frac{dB_s}{d\tau} = b_0 B_s + b_4 B_s A_c^2 + b_5 B_s A_c^4 + \dots, \tag{6.2}$$

where a_2 and b_5 are new parameters, unrelated to the a_2 and b_5 in (4.1), which must be determined.

Equation (6.1) has an equilibrium solution, which is descriptive of Taylor-vortex flow; the first three terms of (6.1) are each of order $a_0^{\frac{3}{2}}$, whereas the quintic term is of order $a_0^{\frac{5}{2}}$. It is *this* feature which implies that a cubic truncation (as in (4.1)) is uniformly valid for a calculation of the equilibrium state (as $a_0 \to 0$), though not necessarily, we hasten to add, for a discussion of stability or instability of that state. With a_0 regarded as a small parameter, the equilibrium solution of (6.1) to order a_0^2 is

$$A_e^2 = (-a_0/a_1) - a_0^2 a_2/a_1^3 + \dots. \tag{6.3}$$

Substitution of this expression in (6.2) yields

$$\frac{dB_s}{d\tau} = B_s\{b_0 - b_4 a_0/a_1 + [b_5 - b_4 a_2/a_1] a_0^2/a_1^2\}, \tag{6.4}$$

so to $O(a_0^2)$, the stability or instability of the Taylor-vortex flow against B_s perturbations depends on the sign of the expression

$$b_{0r} - b_{4r} a_0/a_1 + [b_{5r} - b_{4r} a_2/a_1] a_0^2/a_1^2. \tag{6.5}$$

In order to assess the importance of the term $O(a_0^2)$ in (6.5), it is necessary to know the magnitude of b_{5r} and a_2. Reynolds & Potter (1967) have calculated the Taylor-vortex flow to sixth order in amplitude, and from their data we have estimated that $a_2 \sim -22$ when $a_1 \sim -10$. To estimate b_{5r}, we again consider the approximation $k \to 0$. In this limit b_5 tends to a_2 just as b_4 is known to tend to a_1. Briefly, the argument is as follows. Since (6.4) is linear to B_s, and is the equation of a perturbation which *differs in ζ-phase by* $\frac{1}{2}\pi$ from the Taylor vortex, A_c, equation (6.4) must become equivalent to the equation for an A_s perturbation† to the Taylor-vortex ($B_s \to A_s$) as $k \to 0$. By utilizing 'group' arguments that A_c and A_s in various combinations can only represent shifts in ζ-phase, we can show that the equations corresponding to (4.1), but of quintic order and involving A_c and A_s alone, must be of the form

$$\frac{dA_s}{d\tau} = A_s[a_0 + a_1(A_c^2 + A_s^2) + a_2(A_c^2 + A_s^2)^2 + \ldots], \tag{6.6}$$

together with a similar equation with the c and s suffixes interchanged. Since equation (6.2) must become equivalent to the linearized (in A_s) form of equation (6.6), it follows that $B_s \to A_s$, $b_0 \to a_0$, $b_4 \to a_1$ and $b_5 \to a_2$.

Thus, in the limit $k \to 0$, (6.5), like (5.10), tends to zero. More precisely we may deduce, as we did for $b_{0r} - b_{4r}a_0/a_1$ in equation (5.10), that (6.5) is $O(k^2)$. In the case of $b_{0r} - b_{4r}a_0/a_1$ more exact calculations led to the values of b_{4r}/a_1 given in table 2, and then the values of $(b_{0r} - b_{4r}a_0/a_1)$ given in table 4. For an estimate of b_{5r}, we can at present only (and hopefully) make an intelligent guess. Let us suppose that b_{5r}/a_2 is approximately the same as b_{4r}/a_1. Then a rough lower limit to the magnitude of $(b_{5r} - b_{4r}a_2/a_1)$ is zero. A rough upper limit to its order of magnitude can be obtained by setting $b_{5r} = a_2$, with b_{4r}/a_1 given by table 2. This guess leads to the following:

$$\left.\begin{aligned} |b_{5r} - b_{4r}a_2/a_1| &\simeq \tfrac{1}{20}|a_2| \quad \text{for} \quad m = 1; \\ &\simeq \tfrac{1}{5}|a_2| \quad \text{for} \quad m = 2; \\ &\simeq \tfrac{4}{5}|a_2| \quad \text{for} \quad m = 4. \end{aligned}\right\} \tag{6.7}$$

The sign of the correction term is unknown. Finally, in the neighbourhood of $T = 1820$ to 1835, $a_0 \sim 1$ so with $a_1 \sim -10$ and $a_2 \sim -22$,

$$\left.\begin{aligned} |(b_{5r} - b_{4r}a_2/a_1)a_0^2/a_1^2| &\simeq 0{\cdot}011 \quad \text{for} \quad m = 1, \\ &\simeq 0{\cdot}044 \quad \text{for} \quad m = 2, \\ &\simeq 0{\cdot}18 \quad\;\; \text{for} \quad m = 4. \end{aligned}\right\} \tag{6.8}$$

Insertion of these numbers into the data of table 4, as a correction to $(b_{0r} - b_{4r}a_0/a_1)$, indicates that the value of T at which this expression changes sign alters by about 30 for $m = 1$ and 2, and by a value greater than 30 for $m = 4$. Similar results follow from modification of equations (5.21) and (5.22) by use of (6.8). The corrections are rather small compared to experimental accuracy in relation to T (~ 1700) but are not negligible in relation to $(T'(m) - T_c(0))$, which is of order 120.

A matter related to that of neglecting quintic terms is that of evaluating b_4 and a_1 at $T = 1695$ only. However, as Watson (1960) has argued very cogently in a

† For example, for the amplitude equations (4.1) we would have

$$dA_s/d\tau = a_0A_s + a_1A_c^2A_s \quad \text{and} \quad dB_s/d\tau = b_0B_s + b_4A_c^2B_s;$$

thus $b_0 \to a_0$ and $b_4 \to a_1$ as $k \to 0$ as has been demonstrated by other means.

related context, there is no point (at cubic order) in evaluating these coefficients to better accuracy than $O(1)$. If the calculations were done at other values of T small changes $O(a_0)$ or $O(b_{0r})$ in a_1 and b_{4r} would be introduced,† with resultant changes in the amplitude equations (see (6.3) and (6.4)) of the same order as the corrections due to the quintic terms. Thus, we conclude that in the small-gap model the scheme of calculation of § 5, with its limitation to cubic terms and neglect of other terms of like order, leads to inaccuracies of order 30 to 50 in the evaluation of the critical Taylor number $T'(m)$ for the instability of the Taylor-vortex flow.

Having given *some* justification for the approximations used, we turn to a comparison of the values of m and of $T'(m)$ with those observed experimentally. In the present case, as mentioned earlier, the theory gives a value for $T'(m)$ which is about 8 % above T_c (though this figure could be affected by quintic terms). The weak, monotonic increase of $T'(m)$ with m may or may not be significant; it could be affected by the small-gap approximation and by the neglect of quintic terms. However, there is, on the face of it, a slight preference for $m = 1$ as the most unstable mode.

The experimental data of Coles (1960, 1965) shows that, for an apparatus with $d/R_1 = \frac{1}{7}$, $T'(m)$ is about 55 % above T_c, while $m = 4$. The work of Schwarz, Springett & Donnelly (1964), on the other hand, with $d/R_1 = \frac{1}{19}$ gives *two* values of $T'(m)$, whose significance we shall discuss shortly, but, we note here that a 'weak' mode has its $T'(m)$ about 5 % above T_c‡ and a 'wavy' mode 20 % above both with $m = 1$. The wavy-mode value of 20 % is roughly equivalent to a value of 15 % inferred by Donnelly (1963) using the ion technique, also for $d/R_1 = \frac{1}{19}$. Nissan *et al.* (1963) have reported a $T'(m)$ of about 40 % above T_c, with $m = 1$ for d/R_1 of about $\frac{1}{6}$. The present theory, with its concomitant approximation of $d/R_1 \to 0$, ought to be at its best in comparison with the Chicago experiments of Schwarz *et al.* (1964) and Donnelly (1963), for which d/R_1 is smallest. For reasons which will be clearer later, after a discussion of the *form* taken by the motion *after* instability, the present theoretical result that $[T'(m) - T_c]/T_c$ is about 8 % (a value, furthermore, which is roughly independent of m) cannot necessarily be compared with either the 5 % or the 20 % results of the Chicago group, since the former represents experimentally the initiation of some non-axisymmetric mode and the latter the completion of its evolution towards the wavy-vortex form. Our 'slight preference' for $m = 1$ is in accordance with their work. It is felt that differences between theory and experiment may be attributable to the use of the small-gap approximation and to the neglect of quintic terms. Needless to say, it would be of enormous interest and importance to evaluate this statement quantitatively.

We now turn our attention to the form taken by the flow after the Taylor-vortex flow becomes unstable ($T > T'(m)$). According to the simplified model of § 5, the new motion is the (stable) wavy-vortex mode II (i), described by formulae

† Such changes in a_1, b_4, etc., simply lead to rearrangements of the series expansions for v_{c10}, v_{c11}, etc.

‡ H. A. Snyder and R. B. Lambert (Brown University) have kindly informed us of their related observations of a weak mode with $m = 1$, 2, or 4 for $\eta + 0{\cdot}96$.

(5.17). In the unsimplified mathematical model, the corresponding formulae are given by (4.8), while (4.9) gives the azimuthal velocity perturbation from laminar Couette flow. A major feature that we wish to emphasize is that the boundaries between neighbouring vortices are not plane, because of the presence of both sine and cosine terms in the expansions of the velocity components, as indicated, for example, by (4.9). Such 'wavy' boundaries between neighbouring cells have been observed by Coles (1965), by Schwarz *et al.* (1964), and by Nissan *et al.* (1963), so that this amount of qualitative agreement is established.

There is, however, the embarrassing presence of the 'weak' mode observed by Schwarz *et al.* (1964) at a Taylor number of 5 % above critical ($T_c(0)$); that mode had neighbouring vortices separated by planes. No stable mode of that type has been found according to the present theory. It is possible that the lack of waviness of the boundaries in the 'weak' mode of Schwarz *et al.* (1964) was due to the smallness of the amplitude of the oscillations. To assess this possibility, we consider the relevant particle path equations.

If we go to $O(A_c^2)$ and $O(B_s)$ only in the amplitude expansion for u_z, the axial component of velocity, we obtain

$$u_z = -(\nu/2d)\,[A_c f_{30}(x)\sin\lambda\zeta - 2|B_s|\,|h_{30}(x)|\cos\lambda\zeta\cos k\Phi + A_c^2 m_{31}(x)\sin 2\lambda\zeta + \ldots], \quad (6.9)$$

where $\Phi \equiv \phi - \omega t - \chi(r)$ and $\chi(r)$ is a phase factor. Analysis of our computed data indicates that, to within a few per cent, $|h_{30}|$ is approximately $f_{30} = \lambda^{-1}Df_{20}$ and m_{31} is approximately $Df_{20}/4\lambda$. Thus

$$u_z \sim (\nu/2\lambda d)Df_{20}[A_c\sin\lambda\zeta + \tfrac{1}{4}A_c^2\sin 2\lambda\zeta - 2|B_s|\cos\lambda\zeta\cos k\Phi]. \quad (6.10)$$

Assuming that B_s is small, the term $-2|B_s|\cos\lambda\zeta\cos k\Phi$ in (6.10) will produce a small deviation from the cellular boundaries $\zeta = n\pi/\lambda$ for Taylor vortices. At the position of maximum deviation of the cellular boundaries dz will be zero, so it follows from the particle-path equations

$$\frac{dr}{u_r} = \frac{r d\theta}{u_\theta} = \frac{dz}{u_z} = dt \quad (6.11)$$

that u_z will be zero. Setting $\zeta = n\pi/\lambda + \zeta_1$ and linearizing (6.10) about $\zeta = n\pi/\lambda$, it is found that at the positions of maximum deviation from planar boundaries

$$\zeta_1 = \pm\frac{2|B_s|}{\lambda A_c}[1 + (-1)^{n-1}\tfrac{1}{4}A_c]; \quad (6.12)$$

and $\cos k\Phi = \pm 1$. From (5.17) we have

$$\zeta_1 = \pm\frac{2}{\pi}\left(\frac{\sigma_r - \gamma_r\epsilon}{3\epsilon - 2\sigma_r}\right)^{\frac{1}{2}}\left[1 + \frac{(-1)^{n-1}}{4(-a_1)^{\frac{1}{2}}}\left(\frac{3\epsilon - 2\sigma_r}{3 - 2\gamma_r}\right)^{\frac{1}{2}}\right].$$

At a Taylor number of 1865, which is within the range of the observed 'weak' mode with $m = 1$, the data of tables 3 and 4 yields

$$\zeta_1 = \pm(0{\cdot}07)(1 + (-1)^{n-1}(0{\cdot}1))$$

approximately. Consequently the percentage deviation of a boundary from plane ($2\zeta_1$) is about 14 %. Such a change ought to have been observable, but Schwarz *et al.* make no mention of such large deviations in this Taylor number range. However, for $m = 1$ such deviations might not have been so readily noticed, as they have been for experiments where larger values of m, of order 4 or greater, do occur. Moreover, since, as we shall discuss in the next paragraph, Schwarz *et al.* recorded *large* variations in cell width at a T of 1937, we can infer that boundary waviness must have been present for $T < 1937$. It is possible, therefore, that the weak mode can be interpreted as an incipient form of the wavy-vortex mode. Our critical value of 8 % above $T_c(0)$ certainly lies within the experimental band of 5–20 %.

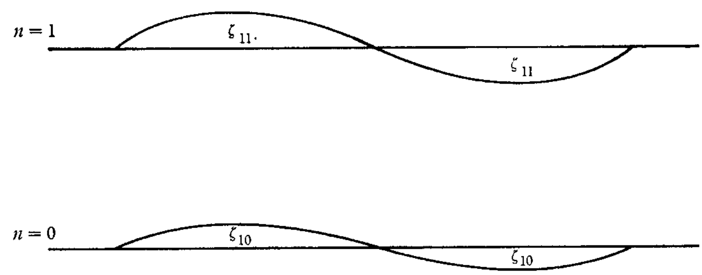

FIGURE 1

We now turn to a discussion of the cell-width variation. It is clear from equation (6.12) that the maximum deviation at successive boundaries ($n = 0$ and $n = 1$) is different, as is shown schematically in figure 1, where ζ_{11} and ζ_{10} are determined from (6.12) using the plus sign with $n = 1$ and 0 respectively. The difference in the cell width at the two extremes is given by

$$2(\zeta_{11} - \zeta_{10}) = \frac{2|B_s|}{\lambda}. \tag{6.13}$$

Compared to the cell width of the Taylor-vortex, the variation in cell width of the wavy vortex, upon substituting for $|B_s|$ from the wavy-vortex solution (5.17) and using (5.12), is

$$\frac{2}{\pi}|B_s| = \frac{2}{\pi}\left(\frac{\sigma_r - \gamma_r \epsilon}{3 - 2\gamma_r}\right)^{\frac{1}{2}} \Big/ (-a_1)^{\frac{1}{2}}. \tag{6.14}$$

For $m = 1$, γ_r is 0·94731 from table 1 and, extrapolating from table 4, we have $\sigma_r - \gamma_r \epsilon = 0{\cdot}05$ approximately for $T = 1937$. Thus with $a_1 = -10$, $2|B_s|/\pi \sim 4\,\%$. Dr K. W. Schwarz and Dr B. E. Springett have kindly informed us that the observations of figure 7 of Schwarz *et al.* (1964) correspond to $T = 1937$; the variation of cell width there is 29 %, so that our value is much too small. The difference may be attributable to the fact that the theory has been taken well beyond its true range of validity.

There is another discrepancy between the above picture and the observations of Coles (1965, p. 400 and plate 1) who noted 'a marked phase shift for alternate cell boundaries'. The model described immediately above, taken to the order shown in (6.10), has the neighbouring cell boundaries in phase. To obtain the shift of phase noted by Coles it would be necessary to include other harmonics in (6.10), in addition to the term A_c^2 retained there; harmonics with ϕ dependence would, it is believed, be able to effect the shift of phase noted by Coles. To the degree to which the present calculations have gone, however, we cannot pursue this matter further.

Turning now to the matter of the torque, we note that, in the small-gap case the theoretical results of Davey (1962), to second-order in amplitude, and of Reynolds & Potter (1967) to sixth order, over-estimate the torque for the Taylor-vortex flow by a large margin when comparison is made with the experiments of Donnelly (1958; see Donnelly & Simon 1960) for d/R_1 of $\frac{1}{19}$. Both Davey and Reynolds & Potter suggested that the discrepancy could be attributed to the occurrence of the wavy modes. This suggestion is given reinforcement by the fact that Davey's torque calculation for the wide-gap case of $R_2 = 2R_1$ shows *much* better agreement with experiment, than does the small-gap case; and, moreover, the wavy-vortex modes of disturbance are known experimentally to occur in the case $R_2 = 2R_1$ only for Taylor numbers of one or two orders of magnitude higher than the critical Taylor number of linearized theory.† If we accept this suggestion, which at least is a reasonable starting point, we are required to accept, or explain, the fact that the theoretically predicted torque is greater than that of experiment, with its associated implication that the wavy vortices *reduce* the torque that would occur otherwise.

For the wavy-vortex flow II (i), it can be inferred from (2.2), (2.5) and (4.9) that the added torque per unit length of this mode (additional to the torque associated with the laminar Couette flow $V(r)$), is

$$\frac{2\pi R_1^3\Omega_0}{d}\nu\rho[F_1'(-\tfrac{1}{2})A_e^2+F_3'(-\tfrac{1}{2})\beta_e^2], \tag{6.15}$$

where A_e and β_e are given by (4.8) and ρ is the density.‡ Substituting for A_e and β_e, the added torque (6.15) can be written in the form

$$\frac{2\pi R_1^3\Omega_0}{d}\nu\rho F_1'(-\tfrac{1}{2})\left\{\frac{-a_0}{a_1}+\left[\Delta-\frac{a_4}{a_1}\right]\beta_e^2\right\}, \tag{6.16}$$

where $\Delta = F_3'(-\frac{1}{2})/F_1'(-\frac{1}{2})$. Of the two parts of (6.16), the part proportional to $(-a_0/a_1)$ is the torque which would be associated with the Taylor-vortex flow (I (i)), if that flow existed as a stable mode at the Taylor number in question. The second part,

$$\frac{2\pi R_1^3\Omega_0\nu\rho}{d}F_1'(-\tfrac{1}{2})\left[\Delta-\frac{a_4}{a_1}\right]\beta_e^2, \tag{6.17}$$

is the excess (or defect, according to sign) of torque over the Taylor-vortex value, due to the fact that the stable mode is the wavy-vortex II (i).

† Private communication of unpublished results from H. Snyder, Brown University, Providence, Rhode Island.

‡ In deriving (6.15) it is not necessary to distinguish between R_1 and R_0.

The question of whether the latter mode has greater or less torque than I (i) can only be resolved by calculation of the factor $[\Delta-(a_4/a_1)]$.

In the limit $k\rightarrow 0$ described in §5, (5.3) and (5.4) indicate that $[\Delta-(a_4/a_1)]$ tends to zero. Consequently a more accurate evaluation of Δ and a_4/a_1 is needed. The former (Δ) can be evaluated from calculations that we have already performed; the results are given in table 5. Notice how small the changes in $F_3'(-\frac{1}{2})$ and Δ are due to changing m, which lends some additional support to the use of the simplified model discussed in §5. To evaluate a_4/a_1, a calculation similar to that required for the evaluation of b_4 is necessary; such a calculation has not been carried out. Thus we cannot say whether the wavy vortex has greater or less torque than the Taylor vortex. Since $\Delta > 2$, we can say that Δ tends to increase the torque but this effect *could* be counteracted if a_4/a_1 were much greater than 2.

$T\ldots$	1694·95		1835	
m	$-F_3'(-\frac{1}{2})$	Δ	$-F_3'(-\frac{1}{2})$	Δ
0	4·68700	2	4·71070	2
1	4·69149	2·0018	4·71609	2·0023
2	4·70463	2·0074	4·73212	2·0091
4	4·75091	2·0272	4·79345	2·0351

TABLE 5. Values of $F_3'(-\frac{1}{2})$ and $\Delta = F_3'(-\frac{1}{2})/F_1'(-\frac{1}{2})$ ($\delta = 0{\cdot}05$, $\mu = 0$, and $\lambda = 3{\cdot}12657$). Scaling is $f_0(0) = 1$, $h_0'(-\frac{1}{2}) = 2{\cdot}8636$ for $T = 1694{\cdot}95$; $f_0(0) = 1$, $h_0'(-\frac{1}{2}) = 2{\cdot}8537$ for $T = 1835$.

Even if $[\Delta-(a_4/a_1)]$ were known accurately, it could still be argued that quartic order terms ($A_e^4, \beta_e^4, A_e^2\beta_e^2$) might be of importance, because $[\Delta-(a_4/a_1)]$ is itself small. The matter must therefore be left in abeyance for the time being, but we note that such a torque calculation would be of great interest.†

So far we have said little about the physical mechanism of instability. Coles (1965, p. 401) has suggested the relevance of mechanisms of general perturbation development in rotating fluids, and has given the empirical formula

$$\left[\frac{R_1}{d}\left(1-\frac{T_c(0)}{T'(m)}\right)\right]^{\frac{1}{2}} \simeq 1{\cdot}5 \tag{6.18}$$

approximately, for the relationship between the critical Taylor number $T'(m)$ of the 'wavy' mode, and the critical value $T_c(0)$ for the Taylor-vortex flow, and the gap ratio d/R_1. When $d/R_1\rightarrow 0$ the above formula (6·18) gives $T'(m)/T_c(0)\rightarrow 1$; whereas, on the other hand, the present theory, which is not a *strict* application of $d/R_1\rightarrow 0$ without qualification, gives a non-zero limiting value for $1-T_c(0)/T'(m)$. This matter requires further investigation.

The ideas expressed by Coles also suggest the relevance of ideas of vortex breakdown (Benjamin 1962), where 'critical' conditions involve the ratio of a swirl velocity to a translational velocity. If a Taylor vortex is regarded as modelled by such a theory, with u_θ as the axial velocity and the pair (u_r, u_z) as the swirl

† W. Debler, of the University of Michigan, has recently informed us that his unpublished experimental work for $\eta = 0{\cdot}95$ on the instability of the Taylor-vortex flow indicates a reduction of torque for the wavy-vortex flow, which apparently is of the Coles' type with $m \geqslant 3$.

velocity, a ratio appears like (6.18) but with d/R_1 replacing R_1/d! On the other hand (6.18) has *some* experimental backing, so that vortex-breakdown concepts may not be relevant.

Meyer (1966) has suggested that the instability, which produces the wavy vortices, arises from an 'Orr–Sommerfeld' instability of the 'jet-like' flow between neighbouring vortices (see Snyder & Lambert 1966, figure 9).

It is possible to give some quantitative backing to Meyer's idea, and to relate it to formula (6.18), as follows: it is known (Davey 1962) that the azimuthal velocity to first order in amplitude is of the form

$$u_\theta = V(r) + A_e R_1 \Omega_1 v_1(r) \cos\lambda\zeta, \tag{6.19}$$

where v_1 ranges between 0 and 1 and

$$A_e^2 = 0{\cdot}3257\{1-(T_c/T)\}. \tag{6.20}$$

Moreover it is known that, in a typical small-gap problem ($d/R_1 = 0{\cdot}05$), the radial and axial velocity components are much smaller (by an order of magnitude) than the perturbation part of (6·19). Let us, therefore, discuss the stability problem of the flow (6.19), regarded as a rectilinear flow, against periodic waves travelling in the azimuth. This poses a stability problem of Orr–Sommerfeld type, with the added complication that u_θ depends on two co-ordinates (r and ζ). Remembering the crucial importance in that theory of inflexion points in the velocity profile, and noting that $\partial^2 u_\theta/\partial\zeta^2$ has zeros at a set of values of ζ, we may reasonably suppose that radial variations are relatively unimportant and therefore approximate (6.19) by a function of ζ only. Thus we have

$$u_\theta = \tfrac{1}{2}R_1\Omega_1 + \tfrac{1}{2}A_e R_1 \Omega_1 \cos\lambda\zeta, \tag{6.21}$$

where the mean and perturbation functions of (6.19) have been replaced by approximate averages.

A characteristic Reynolds number (Re) for this flow can be formed from the velocity difference associated with (6.21), namely $A_e R_1 \Omega_1$, and the length d (since $\lambda \sim \pi$). This yields

$$\begin{aligned} Re &= A_e R_1 \Omega_1 d/\nu \\ &= (0{\cdot}6)(R_1\Omega_1 d/\nu)\{1-(T_c/T)\}^{\frac{1}{2}} \end{aligned} \tag{6.22}$$

approximately by virtue of (6.20). But, in the limit $d/R_1 \to 0$, we have

$$T = (R_1\Omega_1 d/\nu)^2(d/R_1), \tag{6.23}$$

so that (6.22) becomes approximately

$$Re = 25\left[\left(\frac{R_1}{d}\right)\left(1-\frac{T_c}{T}\right)\right]^{\frac{1}{2}}, \tag{6.24}$$

where we have approximated $T^{\frac{1}{2}}$ by 40 because the relevant Taylor numbers are close to 1700. Clearly, if we identify T in (6.24) with $T'(m)$, we have an expression of the form of Coles's (6.18). Furthermore, from Coles's empirical formula (6.18), the critical value of Re against azimuthally-travelling waves is likely to be about 40, a reasonable value for a flow of the type (6.21).

An estimate of the magnitude of the wave-numbers involved can be obtained as follows. An inviscid, two-dimensional stream-function perturbation to (6.21) of the form $\phi(\zeta)\exp\{i\alpha(x-ct)\}$, where x is a co-ordinate in the azimuthal direction, must satisfy

$$(u_\theta - c)(\phi'' - \alpha^2\phi) - u_\theta''\phi = 0, \tag{6.25}$$

dashes denoting derivatives with respect to ζ. A neutral solution for the case of (6.21) satisfies

$$\phi'' + (\lambda^2 - \alpha^2)\phi = 0,$$

with $c = \frac{1}{2}R_1\Omega_1$. If the perturbation is required to have the same wavelength $(2\pi/\lambda)$ along the ζ-axis as the Taylor vortex (as observed for wavy vortices), we have $\alpha = 0$, which gives a *long* wavelength disturbance in the azimuth. This is roughly in accordance with an azimuthal wave-number m which is a small integer such as 1 or 4, since this yields a non-dimensional wave-number α of order md/R_1. This is small in the case under consideration, $d/R_1 \to 0$. The remarks made above are suggestive that Meyer's hypothesis should be pursued in more detail; it is necessary, especially, to justify more completely a study of (6.19) alone and, even more so, of (6.21). We shall not take the matter any further here.

A final point worth making is the following. Since b_4/a_1 is quite different from 1 for $m = 4$, it is possible that b_3/a_1 would be quite different from 3, its limiting value when $k \to 0$. It is possible, therefore, that the deduction we have made, that the Taylor-vortex flow is stable against B_c perturbations may need to be modified. So far we have no additional evidence either way.

7. Conclusions

Subject to four basic assumptions of the model, namely (i) truncation at cubic amplitude terms, (ii) application of the small-gap approximation, both in the classical form and in our estimation of most of the coefficients in the set of ordinary differential equations (4.1), (iii) restriction of the number of basic modes to four, (iv) cylinders infinitely long, we have reached the following conclusions.

(*a*) The Taylor-vortex flow of a given wavelength is unstable against perturbations which have the same axial wave-number, but with an axial phase shift of $\frac{1}{2}\pi$, and which are also periodic in the azimuth. No instability has been found for perturbations with the same axial phase.

(*b*) The critical Taylor number for the instability of the Taylor-vortex flow is about 8 % above the critical value for the occurrence of these Taylor vortices, $T_c(0)$. The experiments which most closely satisfy the assumptions of the theory indicate a value in the range 5–20 %, possibly as low as 5 % if their weak mode is to be interpreted as an incipient form of the wavy-vortex mode (but see under (*d*)).

(*c*) The azimuthal wave-number (m) is not accurately determined by the theory, but there is a *slight* preference for $m = 1$, in agreement with the experiments of Schwarz *et al.* (1964).

(*d*) For Taylor numbers greater than the critical Taylor number at which the Taylor-vortex flow is unstable a new equilibrium flow with wavy boundaries between cells (wavy-vortex), as observed by Coles (1965), Schwarz *et al.* (1964), and others is established. No solution akin to the 'weak' non-axisymmetric mode with plane cellular boundaries described by Schwarz *et al.* was found, though we have suggested that it may be related to the wavy-vortex mode (see (*b*) above and § 6). We note, however, that Schwarz *et al.* (1964) do not report their observations in enough detail for us to make a completely satisfactory evaluation.

(*e*) Whether the torque of the wavy-vortex flow is greater or less than the torque for the Taylor-vortex flow at the same Taylor number has not been determined. Experimental results suggest a reduction.

(*f*) No clear-cut 'physical' description of the instability mechanism for Taylor vortex flow has emerged, though an Orr–Sommerfeld type of instability is possible.

(*g*) In addition to the emergence of the stable 'wavy-vortex' flow in a certain range of Taylor number, the theory also gives a 'spiral mode' (II (iii), equations (4.11) and (5.19)) which has a range of stability, though it cannot be obtained by infinitesimal perturbations. Such a mode, representing a spiral pattern rotating in the azimuthal direction, has not been observed to the authors' knowledge.

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REFERENCES

Benjamin, T. B. 1962 Theory of the vortex breakdown phenomenon. *J. Fluid Mech.* **14**, 593–629.

Bisshopp, F. 1963 Flow between concentric rotating cylinders. A note on the narrow gap approximation. *Technical Report* 51, ONR contract Nonr 562(07), Brown University.

Capriz, G., Ghelardoni, G. & Lombardi, G. 1964 Numerical solution of a stability problem in hydrodynamics. *Estratto da Calcolo*, **1**, 19–48.

Capriz, G., Ghelardoni, G. & Lombardi, G. 1966 Numerical study of the stability problem for Couette flow. *Phys. Fluids*, **9**, 1934–6.

Chandrasekhar, S. 1961 *Hydrodynamic and Hydromagnetic Stability*. Oxford: Clarendon Press.

Coles, D. 1960 Paper presented at the Xth International Congress of Applied Mechanics, Stresa (p. 163). Elsevier.

Coles, D. 1965 Transition in circular Couette flow. *J. Fluid Mech.* **21**, 385–425.

Davey, A. 1962 The growth of Taylor vortices in flow between rotating cylinders. *J. Fluid Mech.* **14**, 336–68.

Di Prima, R. C. 1963 Stability of curved flows. *J. Appl. Mech.* **30**, 486–92.

Donnelly, R. J. 1958 Experiments on the stability of viscous flow between rotating cylinders I. Torque measurements. *Proc. Roy. Soc.* A **246**, 312–25.

DONNELLY, R. J. 1963 Experimental confirmation of the Landau law in Couette flow. *Phys. Rev. Lett.* **10**, 7, 282–4.

DONNELLY, R. J. & SCHWARZ, K. W. 1965 Experiments on the stability of viscous flow between rotating cylinders VI. Finite-amplitude experiments. *Proc. Roy. Soc.* A **283**, 531–49.

DONNELLY, R. J. & SIMON, N. J. 1960 An empirical torque relation for supercritical flow between rotating cylinders. *J. Fluid Mech.* **7**, 401–18.

GROSS, A. G. 1964 Numerical investigation of the stability of Couette flow. Ph.D. Thesis, Rensselaer Polytechnic Institute, Troy, New York.

INCE, E. L. 1956 *Ordinary Differential Equations*. Dover: New York.

KRUEGER, E. R. 1962 The stability of Couette flow and spiral flow. Ph.D. Thesis, Rensselaer Polytechnic Institute, Troy, New York.

KRUEGER, E. R., GROSS, A. G. & DI PRIMA, R. C. 1966 On the relative importance of Taylor-vortex and non-axisymmetric modes in flow between rotating cylinders. *J. Fluid Mech.* **24**, 521–38.

LANDAU, L. D. 1944 On the problem of turbulence. *C.R. Acad. Sci. U.R.S.S.* **44**, 311. See also *Fluid Mechanics*, by L. D. Landau and E. M. Lifshitz. Oxford: Pergamon Press, 1959.

MEYER, K. 1966 A two-dimensional time-dependent numerical study of rotational Couette flow. *Los Alamos Report* LA-3497, Los Alamos, New Mexico.

MINORSKY, N. 1962 *Nonlinear Oscillations*. Princeton, New Jersey: Van Nostrand.

NISSAN, A. H., NARDACCI, J. L. & HO, C. Y. 1963 The onset of different modes of instability for flow between rotating cylinders. *A.I.Ch.E. J.* **9**, 5, 620–4.

REYNOLDS, W. C. & POTTER, M. C. 1967 To be published.

ROBERTS, P. H. 1965 Appendix. The solution of the characteristic-value problems. *Proc. Roy. Soc.* A **283**, 550–6.

SCHULTZ-GRUNOW, F. & HEIN, H. 1956 Beitrag zur Couetteströmung. *Z. Flugwiss.* **4**, 28–30.

SCHWARZ, K. W., SPRINGETT, B. E. & DONNELLY, R. J. 1964 Modes of instability in spiral flow between rotating cylinders. *J. Fluid Mech.* **20**, 281–9.

SEGEL, L. A. 1965 The non-linear interaction of a finite number of disturbances to a layer of fluid heated from below. *J. Fluid Mech.* **21**, 359–84.

SNYDER, H. A. & LAMBERT, R. B. 1966 Harmonic generation in Taylor vortices between rotating cylinders. *J. Fluid Mech.* **26**, 545–62.

SNYDER, H. A. & KARLSSON, S. K. 1965 Nonaxisymmetric modes of secondary flow. *Bull. Am. Phys. Soc.* **10**, 24.

STUART, J. T. 1958 On the non-linear mechanics of hydrodynamic stability. *J. Fluid Mech.* **4**, 1–21.

STUART, J. T. 1960 On the non-linear mechanics of wave disturbances in stable and unstable parallel flows, Part I. *J. Fluid Mech.* **9**, 353–70.

STUART, J. T. 1961 On three-dimensional non-linear effects in the stability of parallel flows. *Advances in Aeronautical Sciences*, vols. 3–4, pp. 121–42. Oxford: Pergamon Press.

STUART, J. T. 1964 Contribution to *Stability of Systems*, Discussion at I. Mech. Eng. May 21/22. Published in 1965, *I. Mech. E. Proc.* **178**, **3** M, 54–5.

TAYLOR, G. I. 1923 Stability of a viscous liquid contained between two rotating cylinders. *Phil. Trans. Roy. Soc.* A **223**, 289–343.

WATSON, J. 1960 On the non-linear mechanics of wave disturbances in stable and unstable parallel flows, Part II. *J. Fluid Mech.* **9**, 371–89.

WHITHAM, G. B. 1963 The Navier–Stokes equations of motion. In *Laminar Boundary Layers* (ed., Rosenhead, L.). Oxford: Clarendon Press.